DIMENSIONS IN RANDOM CONSTRUCTIONS

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We consider random fractals generated by random recursive constructions, prove zero-one laws concerning their dimensions and find their packing and Minkowski dimensions. Also we investigate the packing measure in corresponding dimension. For a class of random distribution functions we prove that their packing and Hausdorff dimensions coincide.

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CONTENTS

CHAPTER 1

Introduction

In this dissertation we consider a general type of random fractal and some dimension properties associated with it. Previously, different authors (see bibliography) have studied the properties of random fractals with respect to the Hausdorff measures defined in the early 20th century. In 1985 Taylor and Tricot in [33] defined packing measures as in some sense dual to the Hausdorff measures. We investigate the packing measure properties of random fractals, the topic that has been discussed in the literature to a much lesser extent. The author considers the results obtained interesting in themselves as with any theoretical mathematical issues.

The random fractals considered are generated by a random recursive construction first defined by Mauldin and Williams in [27]. A simple example of a random fractal is a random *Cantor set* considered in example 7.3. To construct it, we choose 2 numbers independently at random with respect to the uniform distribution from the interval [0,1] and take the left most subinterval and the rightmost subinterval inside whom the procedure is the same up to the scaling. The random fractal is obtained by the procedure analogous to the standard middle third Cantor set.

In general, let $n \in \mathbb{N} \cup \{\infty\}$. Define $\Delta = \{1, 2, \ldots, n\}$, if $n \in \mathbb{N}$, and $\Delta = \mathbb{N}$, if $n = \infty$. The random recursive construction consists of a probability space (Ω, Σ, P) and a family of random compact subsets of $\mathbb{R}^{\mathrm{d}}$ indexed by a tree: $\mathbf{J} = \{J_\sigma | \sigma \in \Delta^* = \bigcup_{\nu=0}^{\infty} \Delta^\nu\}$, where $J_\emptyset = J$ is a fixed seed set, $J =$Cl(Int(J)). These random set constructions must satisfy the following properties:

i. The maps $w \to J_\sigma(w)$ are measurable with respect to Σ.

ii. The sets J_σ, if not empty, are geometrically similar to J,

iii. $J_{\sigma * i}$ is a proper subset of J_σ for all $\sigma \in \Delta^*$ and $i \in \Delta$ provided $J_\sigma \neq \emptyset$,

iv. The construction satisfies a random *open set condition*: if ω and τ are two sequences of the same length, then $\mathrm{Int}(J_\omega) \cap \mathrm{Int}(J_\tau) = \emptyset$, and finally

v. Setting for a finite word $\sigma \in \Delta^*$, $\mathrm{diam}(J_{\sigma*i}) = \mathrm{diam}(J_\sigma)T_{\sigma*i}$, then the random vectors $\tau_\sigma = (T_{\sigma*1}, \ldots, T_{\sigma*n})$ are *conditionally independent* and, provided $J_\sigma \neq \emptyset$, distributed as $(T_1, \ldots, T_n)$.

In [27] the condition on the sequence of random vectors τ_σ is that they are independent and identically distributed. However, they are neither independent, nor identically distributed in many examples in which we would like these results to be applicable. Suppose that there exists $\sigma \in \Delta^*$ such that $P(J_\sigma = \emptyset) > 0$, as in example 7.1, *the Mandelbrot percolation.* If we consider percolation process obtained by partitioning the unit square into 4 equal subsquares so that each of them "survives" independently with probability 1/2, then $P(\tau_1 = \overline{0}) = P(J_1 = \emptyset) + P(\tau_1 = \overline{0}|J_1 \neq \emptyset)P(J_1 \neq \emptyset) = 1/2 + (1/2)^5$. On the other hand, $P(\tau_\emptyset = \overline{0}) = (1/2)^4$. One of the possible ways to correct this is to require that the random vectors τ_σ be conditionally independent, i.e. for any finite set $S \subset \Delta^*$ and any collection of Borel sets $B_s \subset [0,1]^\Delta$, $s \in S$,

$$\prod_{s\in S} P(\tau_s \in B_s | J_s \neq \emptyset) = P\Big(\prod_{s\in S} \tau_s \in \prod_{s\in S} B_s | J_s \neq \emptyset \ \forall s \in S\Big)$$

and τ_σ have the same distribution as $\tau_\emptyset$, provided $J_\sigma \neq \emptyset$, i.e. for any $\sigma \in \Delta^*$ and any Borel set $B \subset \mathbb{R}^{\mathrm{d}}$,

$$P(\tau_\sigma \in B | J_\sigma \neq \emptyset) = P(\tau_\emptyset \in B).$$

We study the random limit set, or fractal, $K(w) = \bigcap_{k=1}^{\infty} \bigcup_{\sigma\in\Delta^k} J_\sigma(w)$, $w \in \Omega$. We note that this setting allows random placement of the sets $J_{\sigma*i}$ within J_σ. Thus these constructions include as a special case the random self-similar sets defined independently by Mauldin and Williams in [27] and by Graf in [10] who first carefully studied them. These last constructions are obtained by choosing the similarity mappings according to some probability distribution and thus may be regarded as random recursive constructions. We also recall that the only interesting case in all of these constructions occurs when there is a positive probability that a nontrivial limit set exists, i.e. when $E\Big[\sum_{i=1}^{n} T_i^0\Big] > 1$, (by convention, $0^0 = 0$); otherwise $K(w)$ is almost surely an empty set or a point. We will assume this condition holds throughout the dissertation.

For the convenience of the reader we also recall the various notions of dimension that we will be concerned with. These dimensions and measures are discussed by Falconer in [4], Mattila in [22], Taylor in [31] and [32], and Taylor and Tricot in [33]. Let $g : [0, +\infty) \to [0, +\infty)$ be gauge function, or a non-decreasing function with $g(0) = 0$. The approximating Hausdorff g-measure, $\mathcal{H}^g_\delta(A)$, of A is defined by

$$\mathcal{H}^g_\delta(A) = \inf\Big\{\sum_{i=1}^{\infty} g(\mathrm{diam}(U_i))|A \subset \bigcup_{i=1}^{\infty} U_i,\ \mathrm{diam}(U_i) < \delta\Big\}$$

and the Hausdorff g-measure, $\mathcal{H}^g(A)$, by

$$\mathcal{H}^g(A) = \lim_{\delta \to 0} \mathcal{H}^g_\delta(A).$$

We write $\mathcal{H}^g_\delta = \mathcal{H}^s_\delta$ and $\mathcal{H}^g = \mathcal{H}^s$ when $0 < s < \infty$ and $g(r) = r^s$, for $r \geq 0$. The Hausdorff dimension, $\dim_H A$, of A is defined by

$$\dim_H A = \inf\{s|\mathcal{H}^s(A) = 0\} = \sup\{s|\mathcal{H}^s(A) = \infty\}.$$

Next we define the g-packing measures. These measures naturally arose in two different areas. They were defined by D. Sullivan in [29] to analyze some problems in dynamics and independently by Taylor and Tricot in [33]. In their paper not only are packing measures and dimensions defined, but the exact gauge function for transient Brownian trajectory is calculated.

Let $A \subset \mathbb{R}^{\mathrm{d}}$ and $\delta > 0$. We say that $\{(x_i, r_i)\}_{i=1}^n$ is a δ-packing of A, if $x_i \in A$, $\delta \geq 2r_i > 0$, and $r_i + r_j < \mathrm{dist}(x_i, x_j)$ for $i,\ j = 1, \ldots, n$, $i \neq j$. Then the closed balls $B(x_i, r_i)$ are disjoint. We first define the packing premeasures, $P^g_{0,\delta}$ and P^g_0, by

$$P^g_{0,\delta}(A) = \sup\Big\{\sum_{i=1}^{n} g(2r_i) \mid \{(x_i, r_i)\}_{i=1}^n \text{ is a } \delta\text{-packing of } A\Big\},$$

$$P_0^g(A) = \lim_{\delta \to 0} P_{0,\delta}^g(A).$$

Since P_0^g is not countably subadditive, one needs a standard modification to get an outer measure out of it. The packing g-measure for $A \subset \mathbb{R}^d$ is defined by

$$\mathcal{P}^g(A) = \inf \Big\{ \sum_{i=1}^{\infty} P_0^g(A_i) \mid A \subset \bigcup_{i=1}^{\infty} A_i \Big\}.$$

Then $\mathcal{P}^g$ is a Borel regular outer measure. When $g(r) = r^s$, we denote $\mathcal{P}^g = \mathcal{P}^s$. In analogy with Hausdorff dimension, the packing dimension can be defined in terms of the packing measures:

$$\dim_P A = \inf\{s | \mathcal{P}^s(A) = 0\} = \sup\{s | \mathcal{P}^s(A) = \infty\}.$$

We note that the two stage definition of packing measure makes it somewhat more technical to handle than Hausdorff measure. In some sense there is no way around this. The complexity of packing measures has been analyzed by Mauldin and Mattila in [24]. For example, it is shown there that the Hausdorff dimension function is a Borel class 2 mapping on $K(\mathbb{R}^d)$, the space of compact sets, whereas the packing dimension function, although measurable with respect to the σ algebra generated by analytic sets, is not Borel measurable.

Finally, we recall the upper and lower Minkowski (or box-counting) dimensions. For K, a bounded subset of $\mathbb{R}^d$, and $\delta > 0$, let $N_\delta(K)$ be the smallest number of open balls of radius δ that are needed to cover K. The upper box-counting dimension, or Minkowski dimension, $\overline{\dim}_B K$, of K is defined by

$$\overline{\dim}_B K = \varlimsup_{\delta \to 0} - \log N_\delta(K)/\log\delta = \varlimsup_{j \to \infty} \log N_{2^{-j}}(K)/(j \log 2),$$

and the lower box-counting dimension, $\underline{\dim}_B K$, by

$$\underline{\dim}_B K = \varliminf_{\delta \to 0} - \log N_\delta(K)/\log\delta = \varliminf_{j \to \infty} \log N_{2^{-j}}(K)/(j \log 2),$$

One very useful fact is that the packing dimension of a set can be calculated from the upper box-counting dimension:

$$\dim_P K = \inf\Big\{ \sup_i \overline{\dim}_B K_i | K \subset \bigcup_{i=1}^{\infty} K_i,\ K_i \in K(\mathbb{R}^{\mathrm{d}}) \Big\}.$$

There are several other relations among dimensions of a set:

$$\dim_H K \leq \dim_P K \leq \overline{\dim}_B K \text{ and } \dim_H K \leq \underline{\dim}_B K \leq \overline{\dim}_B K.$$

Mauldin and Williams determined the Hausdorff dimension of the random limit set $K(w)$, even when n is infinite, as follows. Given $K(w) \neq \emptyset$, the Hausdorff dimension is almost surely α, where $\alpha = \inf\{\beta | E\Big[\sum_{i=1}^{n} T_i^{\beta}\Big] \leq 1\}$. Note that in case n is finite, we have $E\Big[\sum_{i=1}^{n} T_i^{\alpha}\Big] = 1$. Here the expectation E is taken with respect to P. Gatzouras in [8] showed that Minkowski dimension of $K(w)$ coincides with its Hausdorff dimension. In theorem 3.2 we present another short proof of this fact. This proof easily extends in theorem 3.3 onto the random fractals studied by Dryakhlov and Tempelman in [3]. Thus if n is finite, all four of the usual notions of dimension: the upper and lower Minkowski dimensions, the Hausdorff and the packing dimensions, agree. If n is allowed to be infinite, the box-counting dimension and packing dimension may be greater than the Hausdorff dimension even if the recursion is deterministic, see [25], theorem 2.11. In this setting in theorem 3.7 we obtain a formula for the upper Minkowski dimension. It turns out that this dimension is no longer a degenerate random variable, i.e. it does not have to take on a certain value with probability 1. For some classes of random recursive constructions with $n = \infty$ that include random self-similar sets we obtain their packing dimension in theorems 3.8 and 3.11.

The situation regarding the α-dimensional Hausdorff measure of these random fractals is fairly well understood. First, by a zero-one law (see theorems 2.1, 2.2 and the remarks following them) the α-Hausdorff measure of $K(w)$ may be 0, $+\infty$, or positive and finite almost surely. Graf in [10] found that in the case of random self-similar sets the α-Hausdorff measure of $K(w)$ is positive and finite provided the random similarity system is *almost deterministic*, specifically in case (i) $P(\sum_{i=1}^{n} T_i^{\alpha} =$

1) = 1 and (ii)$P(\min_{1\le i\le n} T_i \ge \delta | T_i \ne 0) = 1$ for some $\delta > 0$. Graf et al. in [12] extended this result to general random recursive constructions. The situation is similar for the α-dimensional packing measure. We show in theorem 4.3 that under these same two conditions the α-packing measure is positive and finite. Moreover, Graf in [10] showed that if $P(\sum_{i=1}^{n} T_i^\alpha = 1) < 1$, then the $\mathcal{H}^\alpha$-measure is 0 a.s. Correspondingly, we will prove in theorem 4.6 that the α-packing measure of random self-similar sets in this situation is infinite in case a random strong open set condition is also satisfied.

For many of the cases where the α-Hausdorff measure is 0, the exact Hausdorff dimension function has been determined. Specifically, Graf et al. in [12] have found a gauge function g, so that $0 < \mathcal{H}^g(K(w)) < \infty$ a.s. provided $K(w) \ne \emptyset$. This gauge function can be determined by considering the behavior of the distribution function of the random variable X which is the limit of a certain martingale at infinity. The corresponding situation for the exact packing dimension function is more complicated and largely unsolved. There are only two types of constructions for which the solution has been determined. They are the almost deterministic case mentioned above and those constructions such that the limit set is the image of a subordinator, for example, the zero set in Brownian bridge. Feng and Sha in [6] showed that in this case there is no exact packing function in the following sense. If $\phi(t) = t^\alpha L(t)$ where L satisfies a doubling condition: there is some $c > 0$ such that $L(2t) \le cL(t)$ for small t, then either $\mathcal{P}^\phi(K) = 0$ a.s. or else $\mathcal{P}^\phi(K) = \infty$ a.s., depending on the convergence of some integral. Also Fristedt and Taylor in [7] have found criteria when the image of a general subordinator has the exact packing dimesion and when it does not. Other than these cases, the exact packing measure function problem remains open. Let us comment that Liu in [20] claimed to have the exact packing measure gauge function for a Galton-Watson tree in case the number of offspring is at least 2. However, as we shall show in chapter 5, there is a mistake in the proof that this measure is positive a.s. Along these lines Xiao in [34] proved that there is no exact packing dimension for a branching process in case the number of offspring has a geometric distribution. The exact packing dimension of many other stochastic processes has been investigated, for example, Fristedt and Taylor ([7]), Gatzouras and Lalley ([9]), X. Hu ([15],[16]), and Y. Hu ([18]). In theorem 4.11 we find an upper estimate for the exact packing dimension function for general random recursive constructions by considering the rate of decay of the distribution function of the random variable X at 0. The proofs of all results were discovered by the author under the scientific guidance of my adisor, R.D.Mauldin.

We will use the following notation: $B(x,r)$ is an open ball with center x and radius r, for a finite sequence $\sigma \in \Delta^*$, its length is $|\sigma|$ and $\sigma|_k$ denotes the first k elements of that sequence, $\sigma \prec \tau$ signifies that the sequence τ begins with σ, for $\eta \in \Delta^{\mathbb{N}}$, $f_w(\eta) = \bigcap_{k=1}^{\infty} J_{\eta|_k}(w)$, $l_\sigma(w) = \mathrm{diam}(J_\sigma(w))$, for $\Gamma \subset \Delta^*$, $S_\Gamma^\alpha = \sum_{\sigma\in\Gamma} l_\sigma^\alpha$.

We call $\Gamma \subset \Delta^*$ an antichain if for all $\tau, \sigma \in \Gamma$ $\sigma \not\prec \tau$ and $\tau \not\prec \sigma$. An antichain Γ is maximal, if for all $\eta \in \Delta^{\mathbb{N}}$ there exists a unique $k \in \mathbb{N}$ such that $\eta|_k \in \Gamma$ (we denote $\eta|_k$ by $\eta|_\Gamma$), in other words, a maximal antichain is a cut. Especially useful for us will be antichains of the form $\Gamma_r(w) = \{\sigma \in \Delta^* \mid l_{\sigma|_{|\sigma|-1}} > r,\ l_\sigma \le r\}$. For $\tau \in \Gamma_r$ let $\Gamma_{r,\tau} = \{\sigma \in \Gamma_r \mid \mathrm{dist}(J_\sigma, J_\tau) < r\}$. For $\eta \in \Delta^{\mathbb{N}}$ let $\mathcal{G}_r(\eta, w) = \Gamma_{r,\eta|_{\Gamma_r}}$. These last sets reflect geometric clustering. Without loss of generality, we assume that $\mathrm{diam}(J) = 1$.

Fix a point $a \in \mathbb{R}^d$ with $\mathrm{dist}(a, J) \ge 1$. Number 1 here can be replaced with any other positive number. For any $\tau \in \Delta^*$, denote by $S_\sigma^\tau : \mathbb{R}^{\mathrm{d}} \to \mathbb{R}^{\mathrm{d}}$ a random similarity map such that $S_\sigma^\tau(J_\tau) = J_{\tau*\sigma}$. If $J_\tau = \emptyset$ or $J_{\tau*\sigma} = \emptyset$, then we let $S_\sigma^\tau(\mathbb{R}^{\mathrm{d}}) = a$. For $x \in J_\tau$ and $n \in \mathbb{N}$, consider the random n-orbit of x within J_τ, $O_\tau(x,n) = \bigcup_{\substack{|\sigma|=n \\ J_{\tau*\sigma}\cap K \neq \emptyset}} S_\sigma^\tau(x)$. For $I \subset \mathbb{N}^*$, let $O_\tau(x,I) = \bigcup_{\substack{\sigma\in I \\ J_{\tau*\sigma}\cap K\neq\emptyset}} S_\sigma^\tau(x)$. In case $\tau = \emptyset$, $O_\tau(x,I)$ is denoted by $O(x,I)$, $O_\tau(x,n)$ by $O(x,n)$, and S_σ^τ by S_σ.

CHAPTER 2

Zero-one laws

One of the tools we use for exploring the dimensions and packing measures of random recursive constructions is the zero-one law. First we prove a zero-one law for random self-similar sets.

Theorem 2.1 *For any* $\beta > 0$, $P(\mathcal{P}^\beta(K(w)) = 0 | K(w) \neq \emptyset) = 0$ *or* 1.

Proof: Let $C = \{w | K(w) = \emptyset\}$, $A = \{w | \mathcal{P}^\beta(K(w)) = 0\}$, $B_i = \{w|$ exactly i among $T_1, T_2, \ldots \neq 0\}$, $y = P(A)$, $p_i = P(B_i)$. So, $P(C) < 1$ and $P(C) \leq y$. Suppose that $y < 1$. Since the sets B_i partition the probability space, $P(A) = \sum\limits_{i=0}^{\infty} P(A|B_i)P(B_i)$. Let $K_i(w) = K(w) \cap J_i(w)$. Then $K(w) = \bigcup\limits_{i=1}^{\infty} K_i(w)$, $P(A|B_i) = P(\mathcal{P}^\beta(K(w)) = 0|$ exactly i among $J_1, J_2, \ldots \neq \emptyset) = \prod\limits_{j=1}^{i} P(\mathcal{P}^\beta(K_{l_j}(w)) = 0 | J_{l_j} \neq \emptyset)$ by conditional independence. Now using the fact that the similarity vectors have the same conditional distribution, we obtain $\prod\limits_{j=1}^{i} P(\mathcal{P}^\beta(K_{l_j}(w)) = 0 | J_{l_j} \neq \emptyset) = y^i$. Hence $y = \sum\limits_{i=0}^{\infty} p_i y^i$. Consider the function $f\colon [0,1] \to \mathbf{R}$ defined by the formula $f(x) = p_0 + x(p_1 - 1) + \sum\limits_{i=2}^{\infty} x^i p_i$. So, $f(y) = 0$ and the same considerations show that $f(P(C)) = 0$. Since $\sum\limits_{i=0}^{\infty} p_i = 1$, $f(1) = 0$. Not all of $p_i, i \geq 2$ can equal 0 because $P(C) < 1$. Therefore f'' is positive on the interval and $f(x) < 0$ for all $P(C) < x < 1$. Thus $P(A) = P(C)$, and

$$P(A|\overline{C}) = \frac{P(A \cap \overline{C})}{P(\overline{C})} = \frac{P(A) - P(A \cap C)}{P(\overline{C})} = \frac{P(A) - P(C)}{P(\overline{C})} = 0 \text{ or } 1. \blacksquare$$

Now we consider the case of general random recursive constructions. We prove zero-one law provided the following property is satisfied:

(v′) The random vectors $s_\sigma = (S_1^\sigma, S_2^\sigma, \ldots)$ are conditionally independent and for all $i \in \Delta$, $P(T_i > 0) = 1$.

Theorem 2.2 *Suppose that the construction satisfies only the property* (v$'$). *Then for all* $\beta > 0$, $P(\mathcal{P}^\beta(K) = 0 | K \neq \emptyset) = 0$ *or* 1.

Proof: Since for every $i \in \Delta$, $P(T_i > 0) = 1$, for almost every w the sets J_σ, $\sigma \in \Delta^*$ are non-empty, $K(w)$ is non-empty a.s. and the random vectors s_σ are independent. For $k \in \mathbb{N} \cup \{0\}$, let $\mathcal{E}_k$ be the σ-algebra generated by the maps $w \to s_\tau(w)$ with $|\tau| = k$, and let $\mathcal{E}_k^\infty$ be the σ-algebra generated by the maps $w \mapsto s_\tau(w)$ with $|\tau| \geq k$. Then σ-algebras $\mathcal{E}_i$, $i \in \mathbb{N}$ are independent. Let $\Theta = \bigcap\limits_{k=1}^{\infty} \mathcal{E}_k^\infty$. By Kolmogorov's zero-one law (see [Sh]), any Θ-measurable random variable is degenerate, i.e. admits a certain value with probability 1. For $\sigma \in \mathbb{N}^*$, let $K_\sigma(w) = K(w) \cap J_\sigma(w)$. Fix $k \in \mathbb{N}$. Note that $\mathcal{P}^\beta(K) = 0 \iff \forall |\sigma| = k \; \mathcal{P}^\beta(K_\sigma) = 0$. Hence $\{\mathcal{P}^\beta(K(w)) = 0\} \in \mathcal{E}_k^\infty$ for any $k \in \mathbb{N}$, and for any $\beta > 0$, $P(\mathcal{P}^\beta(K) = 0) = 0$ or 1. ∎

Remark. Similarly one can prove that $P(\mathcal{P}^\alpha(K(w)) < \infty | K(w) \neq \emptyset) = 0$ or 1. The proof of theorems 2.1, 2.2 remains valid for any property (or negation thereof) that holds for the whole construction $K(w)$ if and only if it holds independently for every non-empty offspring. We also can replace the measure $\mathcal{P}^\alpha$ with $\mathcal{P}^g$ where g is a gauge function or with packing premeasure $\mathcal{P}_0^\alpha$, if the number of offspring is finite. Hence $\dim_P K(w)$ is a degenerate random variable, i.e. admits a certain value with probability 0 or 1.

Note that property (v)$'$ even without the condition $P(\forall i \; T_i > 0) = 1$ does not hold for a general random recursive construction because its definition does not account for the placement of the offspring.

Open problems. Is it true that the zero-one law holds for random recursive constructions under the only condition that vectors $s_\sigma(w)$ are conditionally independent? Is it true that the zero-one law holds for every random recursive construction with $n < \infty$?

If the vectors $s_\sigma(w)$ are not conditionally independent, we show in example 7.6 that, in general, it does not hold.

CHAPTER 3

Dimensions of random recursive constructions

Proposition 3.1 *Suppose that a random recursive construction with $n < \infty$, $\mathbf{J}$, satisfies only the properties i, ii, iii from the definition in chapter 1, and for some $\beta > 0$*

1. *for almost every $w \in \Omega$, $\lim_{k\to\infty} \sup_{|\sigma|=k} l_\sigma = 0$ and*

2. $\overline{\lim_{k\to\infty}} \frac{\log E[S_k^\beta]}{k} < 0.$

Then $\overline{\dim}_B K(w) \leq \beta$ a.s.

Proof: Suppose that there exists a set A such that $P(A) > 0$ and for all $w \in A$, $\mathcal{P}_0^\beta(K(w)) > c > 0$. Then for every $\gamma > 0$ we can find $\{(x_j, r_j)\}$ – a γ-packing of $K(w)$ such that $\sum_j (2r_j)^\beta > c$. From this we construct random antichains $\Gamma_\gamma = \{\sigma | x_j \in J_\sigma, l_\sigma < r_j, l_{\sigma|_{|\sigma|-1}} \geq r_j\}$, where $\sigma = \sigma(x)$ is a code for $\{x\} = \bigcap_n J_{\sigma|_n}$. Then $\sum_{\sigma\in\Gamma_\gamma} l^\beta_{\sigma|_{|\sigma|-1}} > c/2^\beta$. Let $|\Gamma| = \min_{\sigma\in\Gamma} |\sigma|$. Since by property 1 for all $w \in A$ $\lim_{\gamma\to 0} |\Gamma_\gamma(w)| = \infty$, we can find a set $B \subset A$ with positive measure on which the divergence is uniform.

Then for all $w \in B$, we have

$$c \leq 2^\beta \overline{\lim_{k\to\infty}} \{\sum_{\sigma\in\Gamma} l^\beta_{\sigma|_{|\sigma|-1}} |\ |\Gamma| \geq k\} \leq 2^\beta \overline{\lim_{k\to\infty}} \sum_{l\geq k} \sum_{|\sigma|=l} l^\beta_{\sigma|_{|\sigma|-1}}.$$

Let $R_k = \sum_{l\geq k} \sum_{|\sigma|=l} l^\beta_{\sigma|_{|\sigma|-1}}$, then $E[R_k] = n \sum_{l\geq k} E[S^\beta_{l-1}]$. By property 2, for all k large enough all terms of the last sum are majorized by the tail of a geometric series with ratio less than 1. Therefore $\lim_{k\to\infty} E[R_k] = 0$ and since R_k is non-increasing, we obtain $\lim_{k\to\infty} R_k = 0$ a.s. which is a contradiction. Hence $\overline{\dim}_B K(w) = \inf\{\beta | \mathcal{P}_0^\beta(K(w)) = 0\} \leq \beta$. ∎

This proposition can be applied to random recursive constructions with $n < \infty$ to find their packing and Minkowski dimensions.

Theorem 3.2 *For a random recursive construction* $\mathbf{J}$ *with* $n < \infty$, $dim_H K(w) = \overline{\dim}_B K(w) = \underline{\dim}_B K(w) = \dim_P K(w)$ *a.s. provided* $K(w) \neq \emptyset$.

Proof: Take $\beta > \alpha$. If we let $p = E\Big[\sum_{i=1}^{n} T_i^{\beta}\Big]$, then $0 \leq p < 1$, and $E[S_k^{\beta}] = p^k$. Thus conditions 1 and 2 of proposition 3.1 are satisfied, and we obtain that $\overline{\dim}_B K(w) \leq \alpha$ a.s. The result now follows from the general facts that $\dim_H K \leq \dim_P K \leq \overline{\dim}_B K$ and $\dim_H K \leq \underline{\dim}_B K \leq \overline{\dim}_B K$. ∎

The same result holds for the random recursive construction with finite memory considered by Dryakhlov and Tempelman in [3]. In their construction $n < \infty$, but it is defined by different properties:

1. J_σ are subsets of a complete metric space $\mathbb{M}$ and for almost every $w \in \Omega$ and $\sigma \in \Delta^*$, $J_\sigma(w) \neq \emptyset$,

2. $\lim_{k\to\infty} \max_{|\sigma|=k} \operatorname{diam}(J_\sigma) = 0$,

3. $J_\sigma \subset J_\eta, \quad$ if $\eta \prec \sigma$,

4. $\operatorname{diam}(J_\sigma(w)) \leq l_\sigma(w)$, for some family of positive random variables $\{l_\sigma | \sigma \in \Delta^*$ satisfying the following conditions for a.e $w \in \Omega$:

   (a) this family is monotone, i.e. $l_{\sigma,p}(\omega) < l_\sigma(\omega)$ for each $\sigma \in \Delta^*$ and each $p \in \Delta$,

   (b) for every $\pi \in \Delta^{\mathbb{N}}$, $\lim_{k\to\infty} l_{[\pi|k]}(w) = 0$, and

   (c) for all $\pi \in \Delta^{\mathbb{N}}$, $\lim_{k\to\infty} \frac{\log l_{[\pi|k+1]}(w)}{\log l_{[\pi|k]}(w)} = 1$.

5. the random vectors $(T_{\sigma*1}, \ldots, T_{\sigma*n}), \sigma \in \Delta^*$, are independent, where $T_{\sigma*i} = l_{\sigma*i}/l_\sigma$,

6. there exists an integer $m \geq 1$ such that for any $\sigma \in \Delta^*$ and any $\eta \in \Delta^{m-1}$ the random vectors $(T_{\sigma*\eta*1}, \ldots, T_{\sigma*\eta*n})$ and $(T_{\eta*1}, \ldots, T_{\eta*n})$ have the same distribution,

7. if neither $\sigma < \eta$ nor $\eta < \sigma$ then for a.e. $w \in \Omega$, $J_\eta(w) \cap J_\sigma(w) \cap K(w) = \emptyset$,

8. there exists $b = b(\omega) > 0$ such that *the Moran index* of the construction $\mathbf{J}$ corresponding to b, $\gamma_w(b) < \infty$

The Moran index has the following definition. Consider a sequence $\pi \in \Delta^{\mathbb{N}}$ and positive numbers r and b. If $bl_{\pi|_m} \geq r$ we define the natural number $k(r, \pi, b) > m$ as follows: $bl_{\pi|_{k(r,\pi,b)+1}} < r \leq bl_{\pi|_{k(r,\pi,b)}}$; if $bl_{\pi|_m} < r$ we put $k(r, \pi, b) = m$. $\gamma_w(b)$ is defined as the minimal number with the following property: for any $x \in \mathbb{M}$, any $\pi \in \Delta^{\mathbb{N}}$ and $k > m$ there exist at most $\gamma_w(b)$ pairwise disjoint sets $J_{\eta^{(t)}|k(bl_{\pi|_k}, \eta^{(t)}, b)}$, $t \in \mathbb{N}$, such that $B(x, bl_{\pi|_k}) \cap \mathrm{Cl}(J_{\eta^{(t)}|k(bl_{\pi|_k}), \eta^{(t)}, b)}) \neq \emptyset$; if such a number $\gamma_w(b)$ does not exist we put $\gamma_w(b) = \infty$.

Theorem 3.3 *In the construction with finite memory by Dryakhlov and Tempelman,* $\dim_P K(w) = \overline{\dim}_B K(w) = \underline{\dim}_B K(w) = \dim_H K(w)$ *a.s.*

Proof: Let $\beta > \dim_H K$ a.s. According to the proofs of theorems 3.1 and 4.1 in [3], $E[S_k^\beta] \leq c\gamma^k$ for $k \geq m$ and some constants $c > 0$, $0 < \gamma < 1$. To finish we apply proposition 3.1. ■

Now we turn to the case when the number of offspring is infinite. Suppose that $n = \infty$ and for the rest of the chapter we assume that properties (i)-(v) from chapter 1 and the following are satisfied:

(vi) it is pointwise finite, i.e. each element of J belongs a.s. to at most finitely many sets J_i, $i \in \mathbb{N}$ (see [25]) and

(vii) J possesses the *neighborhood boundedness property* as introduced in [12]: there exists an $n_0 \in \mathbb{N}$ such that for every $\varepsilon > \mathrm{diam}(J)$, if $J_1, \ldots, J_k$ are non-overlapping sets which are all similar to J with $\mathrm{diam}(J_i) \geq \varepsilon > \mathrm{dist}(J, J_i); i = 1, \ldots, k$, then $k \leq n_0$.

As shown in [12] and [25], there are several different easily verifiable and commonly occurring conditions on the seed set J under which conditions (vi) and (vii) are satisfied, e.g. the cone conditione from [25]. Similarly to Proposition 2.9 in [26], we prove

Lemma 3.4 *For all* $w \in \Omega$, $n \in \mathbb{N}$, *and any two collections of points* $X = \{x_k\}_{k=1}^{\infty}$, $Y = \{y_k\}_{k=1}^{\infty} \subset \bigcup_{|\sigma|=n} J_\sigma$ *such that for all* $\sigma \in \mathbb{N}^n$ *with* $J_\sigma \cap K \neq \emptyset$, $\mathrm{card}(Y \cap J_\sigma) =$

$\text{card}(X \cap J_\sigma) = 1$, *and all* $\sigma \in \mathbb{N}^n$ *with* $J_\sigma \cap K = \emptyset$, $\text{card}(Y \cap J_\sigma) = \text{card}(X \cap J_\sigma) = 0$, *then* $\overline{\dim}_B X = \overline{\dim}_B Y$.

Proof: This equality reduces to the case when $n = 1$ since for every $n > 1$ the collection of sets $\{J_\tau\}$ such that $|\tau|$ is divisible by n forms a random recursive construction again. Next we show that there exists an $M > 0$ such that

$$\forall r > 0 \ \forall z \in \mathbb{R}^d \ \text{card}\{i \in \mathbb{N} | B(z, r) \cap J_i(w) \neq \emptyset \text{ and } l_i(w) \geq r/2\} \leq M$$

Fix $w \in \Omega$, $z \in \mathbb{R}^d$, $r > 0$. Obviously $B(z, r)$ can be covered by 12^d balls of radius $r/6$. Let B_1 be one of them and place inside B_1 a set similar to J. By the neighborhood boundedness property with $\varepsilon = r/2$, we obtain $\text{card}\{i \in \mathbb{N} | B_1 \cap J_i \neq \emptyset \text{ and } l_i \geq r/2\} \leq n_0$. Therefore it suffices to take $M = 12^d n_0$.

Finally take $0 < r \leq 2$, let $I_r = I_r(w) = \{i \in \mathbb{N} | l_i \leq r/2, J_i \cap K \neq \emptyset\}$ and $I'_r = I'_r(w) = \{i \in \mathbb{N} | l_i > r/2, J_i \cap K \neq \emptyset\}$. Then $N_r(Y \cap I_r) \leq N_{r/2}(X \cap I_r)$. Clearly, for any collection of points $Z = \{z_k\}_{k=1}^\infty$, such that for all $i \in \mathbb{N}$ with $J_i \cap K \neq \emptyset$, $\text{card}(Z \cap J_i) = 1$, and for all $i \in \mathbb{N}$ with $J_i \cap K = \emptyset$, $\text{card}(Z \cap J_i) = 0$, we have $N_r(Z \cap I'_r) \leq \text{card}(I'_r)$. On the other hand $N_r(Z \cap I'_r) \geq \text{card}(I'_r)/M$. Hence,

$$N_r(Y) \leq N_{r/2}(X \cap I_r) + N_r(Y \cap I'_r) \leq N_{r/2}(X) + M N_r(X \cap I'_r) \leq (1 + M) N_{r/2}(X).$$

Thus $\overline{\dim}_B Y \leq \overline{\dim}_B X$. ■

Remark. From the proof of lemma 3.4, we see that if for some $x \in J$, $D > 0$, $0 \leq u \leq d$ and for all $0 < r \leq 2$, $N_r(O(x, 1)) \leq D r^{-u}$, then for all $y \in J$, $N_r(O(y, 1)) \leq 2^d (12^d n_0 + 1) D r^{-u}$.

For $\tau \in \mathbb{N}$, let $\overline{\gamma}_\tau = \overline{\dim}_B O_\tau(x, 1)$ for some $x \in J_\tau$ and let $\overline{\gamma} = \sup_{\tau \in \mathbb{N}^*} \overline{\gamma}_\tau$. By lemma 3.4, $\overline{\gamma}_\tau$ does not depend on the choice of $x \in J_\tau$. Suppose additionally that 1-orbits are not too dense, i.e. there exist $A_n > 0$ such that for all $x \in J$, $t > 0$ and $0 < r \leq 2$ we have $E[N_r(O_\tau(x, 1)) \mathbf{1}_{\{\overline{\gamma}_\tau < t\}} | J_\tau \neq \emptyset] < A_{|\tau|} r^{-t}$. This is true, in particular, if for each level one has a fixed non-random vector of similarities so that at each node of that level each entry of the similarity vector is taken either from it or maps $\mathbb{R}^d$ to a.

For $\tau \in \mathbb{N}^q$ let $\mathcal{E}_\tau$ be the σ-algebra generated by the maps $w \mapsto l_{\tau|_i}(w)$, where $1 \leq i \leq q$ and let $\mathbf{J}^\tau = \{S_\sigma^\tau(J) | \sigma \in \mathbb{N}^*\}$ be the construction obtained by pruning the tree to start at node τ. We will assume that similarity vectors are chosen at each level independently and denote the elements associated with $\mathbf{J}^\tau$ by superscript τ.

Lemma 3.5 *For any $x \in J$, $\max\{\dim_H K, \sup_n \overline{\dim}_B O(x,n)\} = \max\{\dim_H K, \overline{\gamma}\}$ a.s.*

Proof: Fix $w \in \Omega$. For any $\tau \in \mathbb{N}^*$, $O_\tau(S_\tau(x),1) \subset O(x,|\tau|+1)$. Therefore $\overline{\gamma}_\tau = \overline{\dim}_B O_\tau(S_\tau(x),1) \leq \overline{\dim}_B O(x,|\tau|+1) \leq \sup_n \overline{\dim}_B O(x,n)$, and $\overline{\gamma} \leq \sup_n \overline{\dim}_B O(x,n)$.

In the opposite direction we will prove using induction on n that for any $t > 0$, if $P(\max\{\dim_H K, \overline{\gamma}\} < t) > 0$, then there exists $B(n) > 0$ such that for all $0 < r \leq 2$, $E[N_r(O(x,n))\mathbf{1}_{\{\overline{\gamma}<t\}}] \leq B(n) r^{-t}$. When $n = 1$, we let $B(1) = A_0$. Suppose that for all $n \leq k$ and for all $0 < r \leq 2$, there exists $B(n) > 0$ such that $E[N_r(O(x,n))\mathbf{1}_{\{\overline{\gamma}<t\}}] \leq B(n) r^{-t}$. To prove the statement for $n = k+1$, fix $r > 0$ and set $I_r(w) = \{\tau \in \mathbb{N}^k | l_\tau(w) < r/2, J_\tau \cap K(w) \neq \emptyset\}$. Then

$$N_r(O(x, I_r \times \mathbb{N})(w)) \leq N_{r/2}(O(x,I_r)(w)) \leq N_{r/2}(O(x,k)(w))$$

and for a fixed $\tau \in \mathbb{N}^k$,

$$E[N_r(O_\tau(S_\tau(x),1))\mathbf{1}_{\tau \notin I_r}\mathbf{1}_{\{\overline{\gamma}<t\}}] = E[E[N_r(O_\tau(S_\tau(x),1))\mathbf{1}_{\tau \notin I_r}\mathbf{1}_{\{\overline{\gamma}<t\}}|\mathcal{E}_\tau]] \leq$$

$$\leq E[\mathbf{1}_{\{r/l_\tau(w) \leq 2\}} E[N_{r/l_\tau(w)}(O^\tau(x,1))\mathbf{1}_{\{\overline{\gamma}^\tau<t\}}]] \leq E[A_k l_\tau(w)^t r^{-t}].$$

Therefore

$$E[N_r(O(x,k+1))\mathbf{1}_{\{\overline{\gamma}<t\}}] \leq E[N_{r/2}(O(x,k))\mathbf{1}_{\{\overline{\gamma}<t\}}] +$$
$$+ E\Big[\sum_{|\tau|=k} N_r(O_\tau(S_\tau(x),1))\mathbf{1}_{\tau \notin I_r}\mathbf{1}_{\{\overline{\gamma}<t\}}\Big] \leq 2^t B(k) r^{-t} + A_k r^{-t} E\Big[\sum_{|\tau|=k} l_\tau^t\Big].$$

Set $B(k+1) = 2^t B(k) + A_k$. ■

Lemma 3.6 *There exists $B > 0$ such that if $t \in \mathbb{R}$ and $P(t \geq \max\{\dim_H K, \overline{\gamma}\}) > 0$, then $E[N_r(K)\mathbf{1}_{\{t \geq \overline{\gamma}\}}] \leq B r^{-t}$ for all $0 < r \leq 2$.*

Proof: Let $p = E\Big[\sum_{i=1}^{\infty} T_i^t\Big] < 1$. Fix $q \in \mathbb{N}$ such that $E\Big[\sum_{|\tau|=q} l_\tau^t\Big] = p^q < 8^{-t-d}/3$. Find $B \geq 4^{t+1}B(q)$, where $B(q)$ is the same as in the proof of Lemma 2, such that $E[N_r(K)] \leq Br^{-t}$ for $1 \leq r \leq 2$.

Now we prove by induction that if $2 \geq r \geq 1/n$, then $E[N_r(K)\mathbf{1}_{\{t\geq\overline{\gamma}\}}] \leq Br^{-t}$. Suppose this inequality holds for n, and $1/(n+1) \leq r < 1/n$. Let $C_{n+1}(w) = \{\tau \in \mathbb{N}^q | l_\tau \leq 1/2(n+1)\}$, $C_{1/4}(w) = \{\tau \in \mathbb{N}^q | l_\tau \geq 1/4\}$. Since $K = \Big(\bigcup_{\tau\in C_{n+1}} K_\tau\Big) \cup \Big(\bigcup_{\tau\in C_{1/4}} K_\tau\Big) \cup \Big(\bigcup_{\tau\in\mathbb{N}^q\setminus(C_{n+1}\cup C_{1/4})} K_\tau\Big)$, we have

$$N_r(K) \leq N_{1/(n+1)}\Big(\bigcup_{\tau\in C_{n+1}} K_\tau\Big) + \sum_{\tau\in\mathbb{N}^q\setminus(C_{n+1}\cup C_{1/4})} N_{1/(n+1)}(K_\tau) + \sum_{\tau\in C_{1/4}} N_r(K_\tau).$$

We note that $N_{1/(n+1)}\Big(\bigcup_{\tau\in C_{n+1}} K_\tau\Big) \leq N_{1/(2(n+1))}(O(x,q))$ because if $B(y_j, 1/(2(n+1)))$ is a collection of balls of radius $1/(2(n+1))$ covering $O(x,q)$, then the balls $B(y_j, 1/(n+1))$ cover $\bigcup_{\tau\in C_{n+1}} K_\tau$, since $\mathrm{diam}(J_\tau) < 1/(2(n+1))$ for all $\tau \in C_{n+1}$. Therefore

$$E[N_r(K)\mathbf{1}_{\{t\geq\overline{\gamma}\}}] \leq B(q)2^t(n+1)^t + E\Big[\sum_{|\tau|=q} N_{1/(n+1)}(K_\tau)\mathbf{1}_{\{l_\tau\in\mathbb{N}^q\setminus(C_{n+1}\cup C_{1/4})\}}\mathbf{1}_{\{t\geq\overline{\gamma}\}}\Big] +$$

$$+E\Big[\sum_{|\tau|=q} N_r(K_\tau)\mathbf{1}_{\{l_\tau\geq 1/4\}}\mathbf{1}_{\{t\geq\overline{\gamma}\}}\Big].$$

To estimate the second term, let $\tau \in \mathbb{N}^q$, then

$$E[N_{1/(n+1)}(K_\tau)\mathbf{1}_{\{1/4>l_\tau>1/2(n+1)\}}\mathbf{1}_{\{t\geq\overline{\gamma}\}}] =$$

$$= E[E[N_{1/(n+1)}(K_\tau)\mathbf{1}_{\{1/4>l_\tau>1/2(n+1)\}}\mathbf{1}_{\{t\geq\overline{\gamma}\}}|\mathcal{E}_\tau]] =$$

$$= E[\mathbf{1}_{\{1/4>l_\tau>1/2(n+1)\}}E[N_{1/(n+1)}(K_\tau)\mathbf{1}_{\{t\geq\overline{\gamma}\}}|\mathcal{E}_\tau]] \leq$$

$$\leq E[\mathbf{1}_{\{1/4>l_\tau(w)>1/2(n+1)\}}E[N_{1/((n+1)l_\tau(w))}(K^\tau)\mathbf{1}_{\{t\geq\overline{\gamma}^\tau\}}]].$$

Since $l_\tau(w) < 1/4 \leq (n/2(n+1))$, we have $1/n \leq 1/(2(n+1)l_\tau(w))$ and by the induction hypothesis and independence of K^τ from $\mathcal{E}_\tau$, $E[N_{1/(n+1)l_\tau(w)}(K^\tau)\mathbf{1}_{\{t\geq\overline{\gamma}^\tau\}}] \leq B(2(n+1)l_\tau(w))^t$. Therefore

$$E\Big[\sum_{|\tau|=q} N_{1/(n+1)}(K_\tau)\mathbf{1}_{\{l_\tau\in\mathbb{N}^q\setminus(C_{n+1}\cup C_{1/4})\}}\Big] \leq E\Big[\sum_{|\tau|=q} B2^t(n+1)^t l_\tau^t\Big] = Bp^q 2^t(n+1)^t.$$

Finally let $\mathcal{E}'_\tau$ be the σ-algebra generated by the maps $w \mapsto l_{\tau|_i}(w)\mathbf{1}_{\{l_\tau(w)\geq 1/4\}}$ with $0 \leq i \leq q$, $\mathbf{J}'$ be an independent copy of our construction with the limit set K' and elements associated with it denoted by superscript $'$. Observe that

$$E\Big[\sum_{|\tau|=q} N_r(K_\tau)\mathbf{1}_{\{l_\tau\geq 1/4\}}\mathbf{1}_{\{t\geq\overline{\gamma}\}}\Big] = \sum_{|\tau|=q} E\Big[\mathbf{1}_{\{l_\tau\geq 1/4\}} E\Big[N_r(K_\tau)\mathbf{1}_{\{t\geq\overline{\gamma}\}}|\mathcal{E}'_\tau\Big]\Big] \leq$$

$$\leq \sum_{|\tau|=q} E\Big[\mathbf{1}_{\{l_\tau\geq 1/4\}} E\Big[N_r(K^\tau)\mathbf{1}_{\{t\geq\overline{\gamma}\}}|\mathcal{E}'_\tau\Big]\Big] = \sum_{|\tau|=q} E\Big[\mathbf{1}_{\{l_\tau\geq 1/4\}} E\Big[N_r(K^\tau)\mathbf{1}_{\{t\geq\overline{\gamma}\}}\Big]\Big] \leq$$

$$\leq 8^d E[N_r(K')\mathbf{1}_{\{t\geq\overline{\gamma}'\}}]E[\mathrm{card}\{\tau\in\mathbb{N}^q|l_\tau\geq 1/4\}] \leq E[N_r(K)\mathbf{1}_{\{t\geq\overline{\gamma}\}}]/3$$

because $E[\mathrm{card}\{\tau \in \mathbb{N}^q|l_\tau \geq 1/4\}] \leq E\Big[\sum_{|\tau|=q}(4l_\tau)^t\Big] \leq 8^{-d}2^{-t}/3 \leq 8^{-d}/3$. Since $r \leq 2/(n+1)$, we obtain

$$E[N_r(K)\mathbf{1}_{\{t\geq\overline{\gamma}\}}] \leq 1.5(B(q)2^t(n+1)^t + Bp^q 2^t(n+1)^t) \leq r^{-t}4^t(1.5B(q)+Bp^q) < Br^{-t}.\blacksquare$$

Theorem 3.7 *If there exist $A_n > 0$ such that for all $x \in J_\tau$, $t > 0$ and $0 < r \leq 2$ we have $E[N_r(O_\tau(x,1))\mathbf{1}_{\{\overline{\gamma}_\tau<t\}}|J_\tau \neq \emptyset] < A_{|\tau|}r^{-t}$, then $\overline{\dim}_B K = \max\{\dim_H K, \overline{\gamma}\}$ a.s. provided $K \neq \emptyset$.*

Proof: Fix $n \in \mathbb{N}$ and consider a collection of points $X = \{x_i\}_{i=1}^\infty \subset K$ such that for all $\sigma \in \mathbb{N}^n$, $J_\sigma \cap K \neq \emptyset \Rightarrow \mathrm{card}(X \cap J_\sigma) = 1$ and $J_\sigma \cap K = \emptyset \Rightarrow \mathrm{card}(X \cap J_\sigma) = 0$. By lemma 3.4, $\overline{\dim}_B X = \overline{\dim}_B O(x,n)$, and therefore $\overline{\dim}_B K \geq \max\{\dim_H K, \sup_{n\in\mathbb{N}} \overline{\dim}_B O(x,n)\}$. By lemma 3.6, $P(\overline{\dim}_B K > \max\{\dim_H K, \overline{\gamma}\}) = 0$. $\blacksquare$

Theorem 3.8 *Suppose that there exists $A > 0$ such that $E[N_r(O(x,1))] < Ar^{-\overline{\gamma}}$ a.s. for all $0 < r \le 2$. Then $\dim_P K = \overline{\dim}_B K = \max\{\dim_H K, \operatorname{ess\,sup} \overline{\dim}_B O(x,1)\}$ a.s. on $K \neq \emptyset$.*

Proof: Since for a random self-similar set γ_τ, $\tau \in \Delta^*$ are conditionally i.i.d., we obtain that if $K(w) \neq \emptyset$, then $\overline{\gamma} = \operatorname{ess\,sup} \overline{\dim}_B O(x,1)$ a.s. To see this, let $z = \operatorname{ess\,sup} \overline{\dim}_B O(x,1)$, then $\operatorname{ess\,sup} \overline{\gamma}_\tau \le z$ for all $\tau \in \Delta^*$ and $\overline{\gamma} = \sup_\tau \overline{\gamma_\tau} \le z$ a.s. If $z = 0$ or $\gamma_\emptyset = z$ a.s., we are done. Otherwise consider $0 < y < z$ such that $1 > b = P(\overline{\dim}_B O(x,1)) \le y > 0$. For all $\tau \in \Delta^*$, $b = P(\overline{\gamma}_\tau > y | J_\tau \neq \emptyset)$. Now we prove that for any $1 > \varepsilon > 0$, $P(\forall \tau\, \overline{\gamma}_\tau \le y \cap K \neq \emptyset) \le \varepsilon P(K \neq \emptyset) + \varepsilon$. Find $m \in \mathbb{N}$ such that $(1-b)^m < \varepsilon$. From [27] it is known that if S_k denotes the number of non-empty offspring on level k, then for almost every $w \in \{K \neq \emptyset\}$, $\lim_{k\to\infty} S_k = \infty$ and for almost every $w \in \{K = \emptyset\}$, $\lim_{k\to\infty} S_k = 0$. Therefore we can find $Z_\varepsilon \subset \Omega$ and $k \in \mathbb{N}$ such that $P(\{K \neq \emptyset\} \ominus \{w | \operatorname{card}(\tau \in \Delta^k \colon J_\tau \neq \emptyset) \ge m\}) < \varepsilon/2$. Now we have

$$P(\forall \tau\, \overline{\gamma}_\tau \le y \cap K \neq \emptyset) \le P(\forall \tau\, \overline{\gamma}_\tau \le y \cap \{w | \operatorname{card}(\tau \in \Delta^k \colon J_\tau \neq \emptyset) \ge m\}) + \varepsilon/2 \le$$

$$\le \sum_{\substack{I \subset \Delta^k \\ \operatorname{card}(I) \ge m}} P(\forall \tau \in I\, \overline{\gamma}_\tau \le y \cap \{\forall \tau \in I\; J_\tau \neq \emptyset\}) + \varepsilon/2 \le$$

$$\le (1-b)^m (P(\{K \neq \emptyset\}) + \varepsilon/2) + \varepsilon/2 \le \varepsilon P(\{K \neq \emptyset\}).$$

Examination of the proofs of lemmas 3.5, 3.6, and theorem 3.7 shows that for every $\tau \in \Delta^*$, $\overline{\dim}_B K_\tau = \max\{\dim_H K, \operatorname{ess\,sup} O(x,1)\}$ provided $K_\tau \neq \emptyset$. Consider a countable cover E_i of $K(w)$ such that $E_i \cap K(w) \neq \emptyset$. Then one of the closures of these sets, $\overline{E_i}$, must have non-empty interior. Therefore there exists $\sigma \in \mathbb{N}^*$ such that $\overline{E_i} \cap K(w) \supset J_\sigma \cap K(w) \neq \emptyset$. Suppose that $t < \max\{\dim_H K, \operatorname{ess\,sup} \overline{\dim}_B O(x,1)\}$. Then $\overline{\dim}_B J_\sigma \cap K > t$ and $\mathcal{P}_0^t(J_\sigma \cap K) = \infty$. From the definition of the packing measure we see that $\mathcal{P}^t(K) = \infty$. ∎

We are unable so far to find the packing dimension in general case. Therefore in the following propositions we make some estimates which sometimes yield the exact answer. For the rest of the chapter we deal with random recursive constructions satisfying property (v′) from chapter 2.

Proposition 3.9 $\max\{\alpha,\beta_1\} \le \dim_P K \le \max\{\alpha,\beta_2\}$ *a.s., where*

$$\beta_1 = \sup\{t | \sum_{\tau\in\mathbb{N}^*} P(\overline{\gamma}_\tau < t) < \infty\} = \inf\{t | \sum_{\tau\in\mathbb{N}^*} P(\overline{\gamma}_\tau < t) = \infty\},$$

$$\beta_2 = \inf\{t | \sum_{\tau\in\mathbb{N}^*} P(\overline{\gamma}_\tau > t) < \infty\} = \sup\{t | \sum_{\tau\in\mathbb{N}^*} P(\overline{\gamma}_\tau > t) = \infty\}.$$

Proof: Let $t > \max\{\alpha,\beta_2\}$. Find $\varepsilon > 0$ such that $t > t-\varepsilon > \max\{\alpha,\beta_2\}$. By the Borel-Cantelli lemma, for almost every w there exist only finitely many τ, so that $\overline{\gamma}_\tau > t-\varepsilon$. Therefore we can find $n(w) \in \mathbb{N}$ such that for every τ whose number is greater than $n(w)$, $\overline{\gamma}_\tau \le t-\varepsilon$. Let $k(w)$ be the minimal word length starting from which $\overline{\gamma}_\tau \le t-\varepsilon$ for all $|\tau| \ge k(w)$. By theorem 3.7, $\overline{\dim}_B J_\tau(w)\cap K(w) \le t-\varepsilon$, and hence for all $|\tau| = k(w)$, $\mathcal{P}_0^t(J_\tau(w)\cap K(w)) = 0$. Consequently, $\mathcal{P}^t(K(w)) = 0$ a.s. The other part can be proved similarly. ■

Proposition 3.10 *Suppose that there exists a sequence of antichains Γ_i with $\lim_i |\Gamma_i| = \infty$ such that for fixed i, $\overline{\gamma}_\tau$ has the same distribution for all $\tau \in \Gamma_i$. Then* $\dim_P K \ge \max\{\alpha, \overline{\lim_i} \operatorname{ess\,sup} \overline{\gamma}_{\tau_i}\}$ *a.s., where τ_i are some nodes from Γ_i.*

Proof: Consider a countable cover E_i of $K(w)$ such that $E_i \cap K(w) \ne \emptyset$. Then one of the closures of these sets, $\overline{E_i}$, must have non-empty interior. Therefore there exists $\sigma \in \mathbb{N}^*$ such that $\overline{E_i}\cap K(w) \supset J_\sigma \cap K(w) \ne \emptyset$. Suppose that $t < \max\{\alpha, \overline{\lim_i} \operatorname{ess\,sup} \overline{\gamma}_{\tau_i}\}$. Then we can find $k \in \mathbb{N}$ such that $k > |\sigma|$ and $\sum_{\tau\in\Gamma_k} P(\overline{\gamma}_\tau > t) = \infty$. Hence, $\overline{\dim}_B J_\sigma \cap K > t$ and $\mathcal{P}_0^t(J_\sigma\cap K) = \infty$. From the definition of the packing measure we see that $\mathcal{P}^t(K) = \infty$. ■

As a consequence of propositions 3.9 and 3.10, we obtain the following

Theorem 3.11 *Suppose that $\overline{\gamma}_\tau$ have the same distribution for words τ of the same length, e.g. we use the same distribution for similarity vectors on each level in the construction, and there exist $A_n > 0$ such that for all $x \in J_\tau$, $t > 0$ and $0 < r \le 2$ we have $E[N_r(O_\tau(x,1))\mathbf{1}_{\{\overline{\gamma}_\tau<t\}} | J_\tau \ne \emptyset] < A_{|\tau|} r^{-t}$. Then* $\dim_P K = \max\{\dim_H K, \overline{\lim_{|\tau|\to\infty}} \operatorname{ess\,sup} \overline{\gamma}_\tau\}$ *a.s.*

Remark. In this case, if $\operatorname{ess\,inf} \overline{\gamma}_\emptyset - \sup_\tau \max\{\alpha, \operatorname{ess\,sup} \overline{\gamma}_\tau\} > 0$, we obtain by theorem 3.7 that $\dim_P K < \operatorname{ess\,inf} \overline{\dim}_B K$ a.s. on $K \neq \emptyset$, and $\overline{\dim}_B K$ may be a non-degenerate random variable. Such construction is produced in example 7.5.

Open problems: What are the packing and lower Minkowski dimensions of a general random recursive construction?

Is theorem 3.7 true without the condition that there exist $A_n > 0$ such that for all $x \in J_\tau$, $t > 0$ and $0 < r \leq 2$ we have $E[N_r(O_\tau(x,1))\mathbf{1}_{\{\overline{\gamma}_\tau < t\}} | J_\tau \neq \emptyset] < A_{|\tau|} r^{-t}$.

CHAPTER 4

Packing measures in random recursive constructions

In this chapter we assume that $n < \infty$ and study the packing measure of the limit set $K(w)$ in its dimension. The results adduced here have been published in [1]. Let $\mathcal{E}_k$ be the σ-algebra generated by the maps $w \to l_\sigma(w)$ where $|\sigma| \leq k$. A basic fact is that the sequence $(S^\alpha_{\Delta^k}, \mathcal{E}_k)$ forms an L^p-bounded martingale for all $p \geq 1$. We denote $X = \lim\limits_{k\to\infty} S^\alpha_{\Delta^k}$. It is known (see [10], [12], [27]) that $E[X] = \mathrm{diam}(J)^\alpha$. Similarly, we define $X_\sigma = \lim\limits_{k\to\infty} \sum\limits_{\tau\in\Delta^k} \prod\limits_{i=1}^{|\tau|} T^\alpha_{\sigma*\tau|_i}$. It is also known that X_σ is distributed as $X/\mathrm{diam}(J)^\alpha$, $E[X_\sigma] = 1$, for $\sigma, \tau \in \Delta^*$ such that $\sigma \not\prec \tau$ and $\tau \not\prec \sigma$, X_σ and X_τ are independent.

Graf et al. in [12] have demonstrated that with each construction one can associate 3 measures, denoted ν_w (the construction measure), μ_w and Q as follows. First, ν_w is determined by setting for a compact set $A \subset \mathbb{R}^\mathrm{d}$

$$\nu_w(A) = \lim_{k\to\infty} \sum_{\substack{\sigma\in\Delta^k \\ J_\sigma\cap A\neq\emptyset}} l^\alpha_\sigma(w) X_\sigma(w).$$

Second, μ_w, a measure on $\Delta^{\mathbb{N}}$, is determined from each set $A(\sigma) = \{\eta \in \Delta^{\mathbb{N}} \mid \sigma \prec \eta\}$, a clopen subset of $\Delta^{\mathbb{N}}$, by

$$\mu_w(A(\sigma)) = l^\alpha_\sigma(w) X_\sigma(w)$$

and μ_w is extended to a Borel measure on $\Delta^{\mathbb{N}}$. Finally, Q is a measure on the product space $\Delta^{\mathbb{N}} \times \Omega$: for a Borel set B, let $B_w = \{\eta \in \Delta^{\mathbb{N}} \mid (\eta, w) \in B\}$. Then

$$Q(B) = \frac{\int \mu_w(B_w) dP(w)}{\mathrm{diam}(J)^\alpha}.$$

Expectations with respect to measures P and Q are connected in the following way (see [12]): if Γ is a map from Ω into the countable set of all maximal antichains such that for each maximal antichain Υ, $\Gamma^{-1}(\Upsilon)$ is in the σ-algebra generated by $\{J_\sigma | \sigma \preceq \Upsilon\}$ and $Y : \Delta^{\mathbb{N}} \times \Omega \to \mathbb{R}$ is a random variable such that $Y(\eta, w) = Y(\eta', w)$ provided $\eta|_{\Gamma(w)} = \eta'|_{\Gamma(w)}$, then

$$E_Q[Y] = \frac{E\Big[\sum_{\sigma\in\Gamma} l_\sigma^\alpha X_\sigma Y(\sigma,\cdot)\Big]}{\operatorname{diam}(J)^\alpha}.$$

In particular, for all $p > 0$ and $\sigma \in \Delta^*$, $E_Q[X_\sigma^p] = E[X_\emptyset^{p+1}] < \infty$.

The next theorem states that under a commonly occurring clustering growth rate, the packing measure of a random self-similar set is almost surely positive. This growth rate condition was studied extensively by Graf et. al in [12] in connection with calculating the exact Hausdorff gauge function.

Theorem 4.1 *Suppose that $K(w)$ satisfies the zero-one law (e.g., it is a random self-similar set) If there exist $C > 0$ and $b \in (0,1)$ such that for every $r > 0$ and all $k \in \mathbf{N}$ $Q(\operatorname{card}(\mathcal{G}_r(\eta, w)) = k) \le Cb^k$, then $P(\mathcal{P}^\alpha(K(w)) > 0 | K(w) \neq \emptyset) = 1$.*

Proof: Let $F = \{w | K(w) \neq \emptyset\}$. By theorem 2.1 it is enough to prove that $P(\mathcal{P}^\alpha(K(w)) > 0 | F) > 0$. Therefore it is enough to find E such that $P(E \cap F) > 0$ and for all $w \in E \cap F$ there exists $K'(w) \subset K(w)$ such that $P^\alpha(K'(w)) > 0$. For any $x \in K(w)$ we can find an $\eta(x) \in \Delta^{\mathbb{N}}$, so that $\{x\} = f_w(\eta)$. Notice that

$$\varliminf_{r\to 0} \frac{\nu_w(B(f_w(\eta), r) \cap K(w))}{r^\alpha} \le \varliminf_{r\to 0} \frac{\sum_{\substack{\sigma\in\Gamma_r \\ J_\sigma \cap B(f_w(\eta), r) \neq \emptyset}} l_\sigma^\alpha X_\sigma}{r^\alpha} \le \varliminf_{r\to 0} \frac{\sum_{\sigma\in\mathcal{G}_r(\eta,w)} l_\sigma^\alpha X_\sigma}{r^\alpha} \le$$

$$\le \varliminf_{r\to 0} \sum_{\sigma\in\mathcal{G}_r(\eta,w)} X_\sigma.$$

We will show the expected value of this last limit is finite. This in turn allows us to apply the density theorem for packing measures (see [22], Theorem 6.11). We estimate

$$E_Q\Big[\varliminf_{r\to 0}\sum_{\sigma\in\mathcal{G}_r(\eta,w)} X_\sigma\Big] \le \varliminf_{r\to 0} E_Q\Big[\sum_{\sigma\in\mathcal{G}_r(\eta,w)} X_\sigma\Big] = \varliminf_{r\to 0}\sum_{\Upsilon}\int_{\{\mathcal{G}_r=\Upsilon\}}\sum_{\sigma\in\Upsilon} X_\sigma dQ \le$$

$$\le \varliminf_{r\to 0}\sum_{\Upsilon}\sum_{\sigma\in\Upsilon}\int_{\Delta^{\mathbb{N}}\times\Omega} X_\sigma \mathbf{1}_{\{\mathcal{G}_r=\Upsilon\}} dQ.$$

For $p > 1$ and $q > 1$ with $1/p + 1/q = 1$, we have

$$\sum_{\Upsilon}\sum_{\sigma\in\Upsilon}\int_{\Delta^{\mathbb{N}}\times\Omega} X_\sigma \mathbf{1}_{\{\mathcal{G}_r=\Upsilon\}} dQ \le$$

$$\le \sum_{\Upsilon}\sum_{\sigma\in\Upsilon} E_Q[X_\sigma^p]^{1/p} E_Q\Big[\mathbf{1}^q_{\{\mathcal{G}_r=\Upsilon\}}\Big]^{1/q} = E[X_\emptyset^{p+1}]^{1/p}\sum_{\Upsilon} Q(\mathcal{G}_r=\Upsilon)^{1/q}\mathrm{card}(\Upsilon) \le$$

$$\le E[X_\emptyset^{p+1}]^{1/p}\sum_{k=1}^{\infty} k^2 Q(\mathrm{card}(\mathcal{G}_r)=k)^{1/q} \le E[X_\emptyset^{p+1}]^{1/p}\sum_{k=1}^{\infty} k^2 C^{1/q} b^{k/q} < M < \infty,$$

for some $M \in \mathbb{R}$.

Hence, for all $\delta > 0$ there exist $A_\delta \subset \Delta^{\mathbb{N}} \times \Omega$ such that $Q(A_\delta) > 1-\delta$ and $M_\delta \in \mathbb{R}$ such that for all $(\eta, w) \in A_\delta$ $\varliminf_{r\to 0}\nu_w(B(f_w(\eta), r)\cap K(w))/r^\alpha \le M_\delta$. Since $Q(A_\delta) = \int \mu_w(A_{\delta w})dP(w)$, we can find a set E_δ such that $P(E_\delta) > 0$, for all $w \in E_\delta$ $\mu_w(A_{\delta w}) > 1-\delta$ and $\nu_w(K(w)) > 0$. Let $K_\delta(w) = f_w(A_{\delta w})$. Then $K_\delta(w) \subset K(w)$ and as $\delta \searrow 0$, $\nu_w(K_\delta(w))$ goes up to $\nu_w(K(w))$. Thus there is some δ' such that $\nu_w(K_{\delta'}(w)) > 0$. We let $K'(w) = K_{\delta'}(w)$ and $E = E_{\delta'}$. ∎

Corollary 4.2 *Suppose J satisfies the neighborhood boundedness property and there exists $\kappa > 0$, such that $E[1/\min T_i^\kappa | T_i > 0] < \infty$. Then $P(\mathcal{P}^\alpha(K(w)) > 0 | K(w) \neq \emptyset) = 1$.*

Proof: Graf et al.([12]) have shown in lemmas 4.4 and 4.6 that this condition implies that the clustering growth rate condition of theorem 4.1 is satisfied. ∎

Remark. The corollary holds for many known examples, e.g. the zero set of the Brownian bridge, Mandelbrot percolation process, etc.

Next, we turn to the *almost deterministic* setting, i.e., the sum of the random reduction ratios is almost surely 1 and the δ condition holds: if the reduction ratio is nonzero it is greater than δ.

Theorem 4.3 *If $P(T_1^\alpha + \cdots + T_n^\alpha = 1) = 1$ and there is some $\delta > 0$ so that $P(T_i \geq \delta | T_i \neq 0)= 1$ for all $1 \leq i \leq n$, then $\mathcal{P}^\alpha(K(w)) \leq (2/\delta)^\alpha < \infty$ a.s.*

Proof: By definition $\mathcal{P}_0^\alpha(K(w)) = \limsup_{\delta\to 0}\{\sum_i |B_i|^\alpha \mid B(x_i, r_i)$ is a δ-packing of $K(w)\}$. For a δ-packing $B(x_i, r_i)$, consider the set $\Gamma_0(w) = \{\sigma \in \Delta^* \mid x_i \in J_\sigma,\ l_\sigma < r_i,\ l_{\sigma|_{|\sigma|-1}} \geq r_i\}$ and extend it to a maximal antichain $\Gamma(w)$. Then $\sum_i |B_i|^\alpha \leq \sum_{\sigma\in\Gamma(w)} (2l_\sigma)^\alpha/\delta^\alpha$. It is known (see Graf([10]), theorem 6.11) that for a maximal antichain Υ, $P\Big(\sum_{\sigma\in\Upsilon} l_\sigma^\alpha = 1\Big) = 1$. Therefore $P\Big(\sum_{\sigma\in\Gamma(w)} l_\sigma^\alpha = 1\Big) = \sum_{\Upsilon\subset\Delta^*} P\Big(\sum_{\sigma\in\Upsilon} l_\sigma^\alpha = 1 \text{ and } \Gamma = \Upsilon\Big) = \sum_{\Upsilon\subset\Delta^*} P(\Gamma = \Upsilon) = 1$. Hence, $\sum_{\sigma\in\Gamma(w)} (2l_\sigma)^\alpha/\delta^\alpha = (2/\delta)^\alpha$ a.s. ■

As mentioned before it is known that under the hypotheses of theorem 4.1, $\mathcal{H}^\alpha(K(w)) > 0$. Since α-packing measure dominates α-Hausdorff measure, we have:

Corollary 4.4 *Suppose that $P(\sum_{i=1}^n T_i^\alpha = 1) = 1$ and for some $\delta > 0$ $P(T_i \geq \delta | T_i \neq 0) = 1$. Then $0 < \mathcal{H}^\alpha(K(w)) \leq \mathcal{P}^\alpha(K(w)) < \infty$ a.s. Moreover, $\mathcal{P}^\alpha(K(w))$ and $\nu_w(K(w))$ are absolutely continuous with respect to each other given that $K(w) \neq \emptyset$.*

Now, we show that if there is enough randomness in the reduction ratios and a random *strong open set condition* holds then the α-packing measure is infinite.

Definition 4.5 *The construction satisfies a (random) strong open set condition if there are ρ_0, $p_0 > 0$ such that $P(\exists x \in K(w) \cap J_\sigma$ and $\mathrm{dist}(x, \partial J_\sigma) \geq \rho_0 l_\sigma \mid J_\sigma \neq \emptyset) \geq p_0$.*

Theorem 4.6 *Suppose that $K(w)$ satisfies the zero-one law (e.g., it is a random self-similar set) If $P(T_1^\alpha + \cdots + T_n^\alpha = 1) < 1$ and the construction satisfies the random strong open set condition, then $P(\mathcal{P}^\alpha(K(w)) = \infty | K(w) \neq \emptyset) = 1$.*

Proof: First, we deal with the α-packing premeasure. Note that $X = \sum_{i=1}^{n} T_i^\alpha X_i$ which implies that $\operatorname{ess\,inf} X = 0$ and there exist $\epsilon, \kappa > 0$ such that for all sufficiently large k, $P(S^\alpha_{\Delta^k} < 1 - \kappa) > \epsilon$.

According to the strong open set condition there exists $\rho > 0$, perhaps a smaller $\epsilon > 0$ and Z_ϵ such that $P(Z_\epsilon) > 1 - \epsilon/2$ and for all $w \in Z_\epsilon$ there is $\tau \in \Delta^*$ such that $\operatorname{dist}(J_\tau, \partial J) > \rho$ and $J_\tau \cap K(w) \neq \emptyset$. Therefore we can find $k_0 \in \mathbf{N}$ such that for all $k \geq k_0$, $P(\{\sum_{|\sigma|=k} T^\alpha_\sigma < 1-\kappa\} \cap \{\exists \tau, |\tau| = k \colon \operatorname{dist}(J_\tau, \partial J) \geq \rho \text{ and } J_\tau \cap K(w) \neq \emptyset\}) > 0$ and that event (which will be denoted by A) is in the σ-algebra $\mathcal{E}'_k$ generated by the maps $w \to J_\sigma(w)$ where $|\sigma| \leq k$.

Now let $h_m = \sup\{S^\alpha_\Gamma | \Gamma \text{ is an antichain, } \Gamma \neq \{\emptyset\}, \forall \sigma \in \Gamma\ |\sigma| \leq m, \text{ and } \exists x \in J_\sigma \cap K(w) \colon \operatorname{dist}(x, \partial J_\sigma) \geq \rho l_\sigma\}$. Obviously, $h_{m+1} \geq h_m$. Let $h = \lim_{m\to\infty} h_m$. Then it is easy to see that

$$h_{m+1} = \sum_{i=1}^{n} T_i^\alpha \max(\mathbf{1}_{\{\exists\, x \in K(w)\cap J_i \colon \operatorname{dist}(x,\partial J_i) \geq \rho l_i\}}, h^i_m).$$

Therefore

$$h = \sum_{i=1}^{n} T_i^\alpha \max(\mathbf{1}_{\{\exists\, x \in K(w)\cap J_i \colon \operatorname{dist}(x,\partial J_i) \geq \rho l_i\}}, h^i),$$

where h^σ is like h but the supremum is taken over antichains whose elements properly extend σ.

Taking the expected values of both sides, we obtain

$$\int h dP = \sum_{i=1}^{n} E[T_i^\alpha \max(\mathbf{1}_{\{\exists\, x \in K(w)\cap J_i \colon \operatorname{dist}(x,\partial J_i) \geq \rho l_i\}}, h^i)].$$

By independence,

$$\int h dP = \sum_{i=1}^{n} E[T_i^\alpha] E[\max(\mathbf{1}_{\{\exists\, x \in K(w)\cap J_i \colon \operatorname{dist}(x,\partial J_i) \geq \rho l_i\}}, h^i)] =$$

$$= E[\max(\mathbf{1}_{\{\exists\, x \in K(w)\cap J_1 \colon \operatorname{dist}(x,\partial J_1) \geq \rho l_1\}}, h^1)] E\Big[\sum_{i=1}^{n} T_i^\alpha\Big] =$$

$$= E[\max(\mathbf{1}_{\{\exists\, x \in K(w)\cap J:\ \mathrm{dist}(x,\partial J)\geq\rho\}}, h)].$$

From this we obtain

$$\int h\mathbf{1}_{\{\exists\, x\in K(w)\cap J:\ \mathrm{dist}(x,\partial J)\geq\rho\}}dP + \int h\mathbf{1}_{\{\nexists\, x\in K(w)\cap J:\ \mathrm{dist}(x,\partial J)\geq\rho\}}dP =$$

$$= \int \max(1,h)\mathbf{1}_{\{\exists\, x\in K(w)\cap J:\ \mathrm{dist}(x,\partial J)\geq\rho\}}dP + \int h\mathbf{1}_{\{\nexists\, x\in K(w)\cap J:\ \mathrm{dist}(x,\partial J)\geq\rho\}}dP$$

Thus $h \geq 1$ a.s. on the set $B = \{w \mid \exists\, x \in K(w)\cap J\colon\ \mathrm{dist}(x,\partial J) \geq \rho\}$, for all $\sigma \in \Delta^*$, $h^\sigma \geq 1$ a.s. on the set $\{w \mid \exists\, x \in K(w)\cap J_\sigma\colon\ \mathrm{dist}(x,\partial J_\sigma) \geq \rho l_\sigma\}$, and $h = \sum\limits_{|\sigma|=k} l_\sigma^\alpha h^\sigma$ a.s.

Let $\zeta = \operatorname*{ess\,inf}_B h \geq 1$. Suppose that $\zeta \in \mathbb{R}$. Then for all $k \geq k_0$ we have $\sum\limits_{|\sigma|=k} l_\sigma^\alpha h^\sigma \geq \zeta$ for a.e. $w \in B$. Let $C_\sigma = \{w|h^\sigma < \zeta+\kappa\}$. These events are independent of each other and of the σ-algebra $\mathcal{E}'_k$. Hence, $P(\bigcap\limits_{|\sigma|=k} C_\sigma \cap A) > 0$ and we obtain that $(\zeta+\kappa)(1-\kappa) > \zeta$ holds with positive probability which is a contradiction. This shows that $P(\mathcal{P}_0^\alpha(K(w)) = \infty | K(w) \neq \emptyset) > 0$, and hence equals 1 by the zero-one law.

Now let $\{E_i\}_{i=1}^\infty$ be an arbitrary cover of $K(w)$ by closed sets such that $E_i \cap K(w) \neq \emptyset$. Since $K(w) \subset \mathbb{R}^{\mathrm{d}}$ is compact, one of the E_i's must have a non-empty interior, and therefore there exists σ such that $K(w)\cap E_i \supset J_\sigma \cap K(w) \neq \emptyset$. As it has already been proved, $\mathcal{P}_0^\alpha(K(w)\cap J_\sigma) = \infty$ a.s. on $K(w)\cap J_\sigma \neq \emptyset$. The result now follows from the definition of the packing measure. ■

Now we improve the estimates obtained by considering packing measures with respect to a gauge function $\phi(t) = t^\alpha g(t)$. For $k \in \mathbb{N}$ define random variables T_k, l_k on $\Delta^{\mathbb{N}} \times \Omega$ by $T_k(\eta, w) = T_{\eta|_k}(w)$ and $l_k(\eta,w) = l_{\eta|_k}(w)$ respectively. Fix $c > E_Q[|\log T_1|]$ and let $N = e^c$.

Lemma 4.7 *Let $\{Y_i\}_{i\geq 1}$ be a sequence of i.i.d. random variables with expectation 0 and finite fourth moment and let $k \in \mathbb{N}$. Then for all $m \in \mathbb{R}$ there exists an $\tilde{M} > 0$ such that $\sum\limits_{j=k+1}^{\infty} Q(\sum\limits_{i=1}^{j} Y_i < (k+1-j)c + m) < \tilde{M} < \infty$.*

Proof: The proof of lemma 4.7 is similar to the proof of Cantelli theorem ([28], p.388). Let $S_j = \sum\limits_{i=1}^{j} Y_i$. According to this proof, $\sum\limits_{j=1}^{\infty} P(|S_j/j| \geq c/2) < \infty$. Note that $\lim\limits_{j\to\infty}((k+1-j)c+m)/j = -c$. The result follows. ■

In making upper estimates of the Hausdorff measure with respect to a gauge function Graf et al.([12]) used antichains consisting of the cells on the same level in the construction. Here to get upper estimates of the packing measure with respect to a gauge function, we use antichains consisting of the cells of comparable size. Lemma 4.8 gives an estimate on the number of such cells. One can replace the number 2 in the lemma with any number greater than or equal to 1.

Lemma 4.8 *There is an $M > 0$ such that for all $k \in \mathbb{N}$, $E[\mathrm{card}\{\sigma | N^{-k-1} < 2l_\sigma \leq N^{-k}\}] \leq MN^{k\alpha}$.*

Proof: Fix k and set

$$A = N^{-k\alpha} E[\mathrm{card}\{\sigma | N^{-k-1} < 2l_\sigma \leq N^{-k}\}] = N^{-k\alpha} \sum_{j=1}^{\infty} \sum_{|\sigma|=j} E\Big[\mathbf{1}_{\{N^{-k-1} < 2l_\sigma \leq N^{-k}\}}\Big].$$

Since X_σ and $\mathbf{1}_{\{N^{-k-1} < 2l_\sigma \leq N^{-k}\}}$ are independent and $E[X_\sigma] = 1$, we have

$$A = \sum_{j=1}^{\infty} N^{-k\alpha} E\Big[\sum_{|\sigma|=j} l_\sigma^\alpha X_\sigma l_\sigma^{-\alpha} \mathbf{1}_{\{N^{-k-1} < 2l_\sigma \leq N^{-k}\}}\Big] =$$

$$= \sum_{j=1}^{\infty} N^{-k\alpha} E_Q\Big[l_j^{-\alpha} \mathbf{1}_{\{N^{-k-1} < 2l_j \leq N^{-k}\}}\Big] \leq \sum_{j=1}^{\infty} (2N)^\alpha E_Q\Big[\mathbf{1}_{\{N^{-k-1} < 2l_j \leq N^{-k}\}}\Big] =$$

$$= \sum_{j=1}^{\infty} (2N)^\alpha Q(kc \leq |\log l_j| - \log 2 < (k+1)c) \leq (2N)^\alpha \sum_{j=1}^{k} Q(|\log l_j| \geq kc) +$$

$$+ (2N)^\alpha \sum_{j=k+1}^{\infty} Q\Big(\sum_{i=1}^{j} |\log T_i| < (k+1)c + \log 2\Big) \leq (2N)^\alpha (k b_1 c_2^k + \tilde{M}),$$

where the first sum is estimated using inequality (3.30) from the article of Graf et al.([12]) and the second sum is estimated by lemma 4.7 with $Y_i = |\log T_i| - c$ and $m = \log 2$. Note that $E_Q(Y_i^4) = E_P\Big[\sum\limits_{i=1}^{n} T_i^\alpha (|\log T_i| - c)^4\Big] < \infty$, because for any $\alpha > 0$

and any $k \in \{0,1,2,3,4\}$ the function $x^\alpha \log^k x$ is bounded on $[0,1]$. The numbers $b_1 > 0$ and $c_2 \in (0,1)$ depend only on the construction. The result follows. ■

In the proofs and statements of the next two lemmas, we suppose a number M has been fixed such that lemma 4.8 holds.

Lemma 4.9 *Let $\lambda > 0$ and set $M_k(w) = \mathrm{card}\{\sigma | N^{-k-1} < 2l_{\sigma|_{|\sigma|-1}} \le N^{-k}, l_\sigma^\alpha X_\sigma < \lambda\phi(2l_{\sigma|_{|\sigma|-1}})\}$. Then $E[M_k] \le nMN^{k\alpha}P(X < n(2N)^\alpha \lambda g(N^{-k}))$.*

Proof: Since the gauge function $\phi(t)$ is increasing, we obtain

$$E[M_k] = E\Big[\sum_\sigma \mathbf{1}_{\{l_\sigma^\alpha X_\sigma < \lambda\phi(2l_{\sigma|_{|\sigma|-1}})\}\cap\{N^{-k-1}<2l_{\sigma|_{|\sigma|-1}}\le N^{-k}\}}\Big] \le$$

$$\le E\Big[\sum_\sigma \mathbf{1}_{\{T_\sigma^\alpha X_\sigma < (2N)^\alpha \lambda g(N^{-k})\}}\mathbf{1}_{\{N^{-k-1}<2l_{\sigma|_{|\sigma|-1}}\le N^{-k}\}}\Big] =$$

$$= \sum_\tau E\Big[\mathbf{1}_{\{N^{-k-1}<2l_\tau\le N^{-k}\}}\sum_{i=1}^n \mathbf{1}_{\{T_{\tau*i}^\alpha X_{\tau*i} < (2N)^\alpha\lambda g(N^{-k})\}}\Big].$$

The value of the last sum does not exceed n, and it may differ from 0 only on the set $\{w | X_\tau(w) < n(2N)^\alpha \lambda g(N^{-k})\}$. We can continue as follows:

$$E[M_k] \le n\sum_\tau E\Big[\mathbf{1}_{\{N^{-k-1}<2l_\tau\le N^{-k}\}}\mathbf{1}_{\{X_\tau < n(2N)^\alpha\lambda g(N^{-k})\}}\Big] \le$$

$$\le nP(X < n(2N)^\alpha \lambda g(N^{-k}))E[\mathrm{card}\{\tau | N^{-k-1} < 2l_\tau \le N^{-k}\}] \le$$

$$\le nMN^{k\alpha}P(X < n(2N)^\alpha\lambda g(N^{-k})).■$$

The next lemma tells us how often the size of a cell can drop significantly from level to level.

Lemma 4.10 *For $k_0, k \in \mathbb{N}$ let $M_k^{(k_0)} = \mathrm{card}\{\sigma | N^{-k-1} < 2l_{\sigma|_{|\sigma|-1}} \le N^{-k}, 0 < l_\sigma < N^{-k_0}\}$. Then for any $\zeta > 0$, $E[M_k^{(k_0)}] \le E[1/\min_{1\le i\le n} T_i^\zeta | T_i > 0]2^\zeta nMN^{k\alpha}N^{\zeta(k+1-k_0)}$.*

Proof:

$$E\Big[M_k^{(k_0)}\Big] \le E\Big[\sum_\sigma \mathbf{1}_{\{0<T_\sigma<2N^{k+1-k_0}\}}\mathbf{1}_{\{N^{-k-1}<2l_{\sigma|_{|\sigma|-1}}\le N^{-k}\}}\Big] =$$

$$= \sum_{\sigma} P(0 < T_\sigma < 2N^{k+1-k_0}) E\Big[\mathbf{1}_{\{N^{-k-1} < 2l_{\sigma|_{|\sigma|-1}} \le N^{-k}\}}\Big] =$$

$$= \sum_{\tau} E\Big[\mathbf{1}_{\{N^{-k-1} < 2l_\tau \le N^{-k}\}}\Big] \sum_{i=1}^{n} P(0 < T_{\tau * i} < 2N^{k+1-k_0}) \le$$

$$\le nP\Big(\min_{1\le i\le n} T_i < 2N^{k+1-k_0} | T_i > 0\Big) E[\text{card}\{\tau | N^{-k-1} < 2l_\tau \le N^{-k}\}],$$

using lemma 4.8 and Chebyshev's inequality,

$$\le E[1/\min_{1\le i\le n} T_i^\zeta | T_i > 0] 2^\zeta n M N^{k\alpha} N^{\zeta(k+1-k_0)}. \blacksquare$$

Theorem 4.11 *(upper bound). Suppose that* $E\Big[1/\min_{1\le i\le n} T_i^\zeta | T_i > 0\Big] < \infty$ *for some* $\zeta > 0$*, then*

1. *If* $P(X < a) \le Ca^\beta$ *as* $a \to 0$ *and* $\phi(t) = t^\alpha g(t)$ *is an arbitrary gauge function, then* $\int_{0^+} \frac{g^{\beta+1}(s)}{s} ds < +\infty$ *implies* $P(\mathcal{P}^\phi(K(w)) = 0 | K(w) \ne \emptyset) = 1$.
2. *If* $r = \liminf_{a\to 0} -a^{-1/\beta} \log P(X < a) < \infty$*, then for* $\phi(t) = t^\alpha |\log|\log t||^\beta = t^\alpha g(t)$*,* $P(\mathcal{P}^\phi(K(w)) < \infty | K(w) \ne \emptyset) = 1$.

Proof: Fix an arbitrary $\lambda > 0$. For $\delta > 0$, choose $k_0 \in \mathbb{N}$ such that $N^{-k_0} \ge \delta > N^{-k_0-1}$. Consider a (random) δ-packing of $K(w)$ consisting of balls $B_i(x_i, r_i)$. Build an antichain $\Gamma = \{\sigma | J_\sigma(w) \ni x_i$ for some i and $l_\sigma(w) < r_i, l_{\sigma|_{|\sigma|-1}}(w) \ge r_i\}$. Then $\sum \phi(|B_i|) \le \sum_{\sigma\in\Gamma} \phi(2l_{\sigma|_{|\sigma|-1}})$ and certainly we have

$$P^\phi_{0,\delta}(K(w)) \le \sup\Big\{\sum_{\sigma\in\Gamma} \phi(2l_{\sigma|_{|\sigma|-1}}) | \Gamma \text{ is an antichain}, \forall \sigma \in \Gamma\ 0 < l_\sigma < N^{-k_0}\Big\}.$$

For such an antichain Γ, let $\Gamma_1 = \{\sigma \in \Gamma | l_\sigma^\alpha X_\sigma \ge \lambda\phi(2l_{\sigma|_{|\sigma|-1}})\}$, $\Gamma_2 = \Gamma \setminus \Gamma_1$. Then $\sum_{\sigma\in\Gamma_1} \phi(2l_{\sigma|_{|\sigma|-1}}) \le \lambda^{-1} \sum_{\sigma\in\Gamma_1} l_\sigma^\alpha X_\sigma \le \lambda^{-1} X$ and using the terminology from lemma 4.9, $\sum_{\sigma\in\Gamma_2} \phi(2l_{\sigma|_{|\sigma|-1}}) \le \sum_{k=1}^{[\log k_0]} M_k^{(k_0)}(w)\phi(N^{-k}) + \sum_{k=[\log k_0]}^{\infty} M_k(w)\phi(N^{-k})$. Thus $\mathcal{P}^\phi_{0,\delta}(K(w)) \le \lambda^{-1}X(w) + \sum_{k=1}^{[\log k_0]} M_k^{(k_0)}(w)\phi(N^{-k}) + \sum_{k\ge[\log k_0]} M_k(w)\phi(N^{-k})$.

Since $P^{\phi}_{0,\delta}(K(w))$ decreases as $\delta \searrow 0$, we obtain by lemmas 4.9 and 4.10 that

$$E[P^{\phi}_{0}(K(w))] = E[\liminf_{\delta\to 0} P^{\phi}_{0,\delta}(K(w))] \leq \liminf_{\delta\to 0} E[P^{\phi}_{0,\delta}(K(w))] \leq$$

$$\leq \liminf_{k_0\to\infty} E\Big[\lambda^{-1}X + \sum_{k=1}^{[\log k_0]} M_k^{k_0}(w)\phi(N^{-k}) + \sum_{k\geq[\log k_0]} M_k(w)\phi(N^{-k})\Big] \leq$$

$$\leq \liminf_{k_0\to\infty} \Big[\lambda^{-1} + nME\Big[1/\min_{1\leq i\leq n} T_i^{\zeta}|T_i>0\Big] \sum_{k=1}^{[\log k_0]} N^{\zeta(k+1-k_0)}g(N^{-k}) +$$

$$+nM \sum_{k\geq[\log k_0]} g(N^{-k})P(X < n(2N)^{\alpha}\lambda g(N^{-k}))\Big].$$

In case 1 we observe that if $\int\limits_{0^+} \frac{g^{\beta+1}(s)}{s}ds < +\infty$ and we set $t = \log_N s$, then $c^{-1}\int\limits_{-\infty} g^{\beta+1}(N^t)dt < \infty$, therefore $\sum\limits_{k=1}^{\infty} g^{\beta+1}(N^{-k}) < +\infty$ and the set $\{g(N^{-k})\}_{k=1}^{\infty}$ is bounded. Hence, for all $\lambda > 0$ $E[P^{\phi}_0(K(w))] \leq \lambda^{-1}$. Thus $P^{\phi}_0(K(w)) = 0$ a. s., and therefore $P(\mathcal{P}^{\phi}(K(w)) = 0|K(w) \neq \emptyset) = 1$.

In case 2 let $0 < t < r$. Then by the definition of r, there is some $C_t > 0$ such that for all $a > 0$, $P(X < a) \leq C_t e^{-ta^{1/\beta}}$ so that $\beta \leq 0$, the set $\{g(N^{-k})\}_{k=1}^{\infty}$ is also bounded and therefore the limit of the sum over first $[\log k_0]$ terms is 0. The tail sum $\leq \sum\limits_{k\geq[\log k_0]} P(X < \lambda(2N)^{\alpha}(\log k)^{\beta}) \leq C_t \sum\limits_{k\geq[\log k_0]} (\log k)^{\beta} k^{-t(\lambda(2N)^{\alpha})^{1/\beta}}$. If $t(\lambda(2N)^{\alpha})^{1/\beta} > 1$, this is the tail of a convergent series. Hence, $P(\mathcal{P}^{\phi}(K(w)) < \infty|K(w) \neq \emptyset) = 1$. ■

Based on the articles of Xiao [34], Liu [20] and examples that follow we conjecture that there is a corresponding lower bound result:

Open problems:

Is it true that every random recursive construction fits into case 1 or case 2 of theorem 4.11?

(Lower bound) In the setting of theorem 4.11, is it true that

1. If $P(X < a) \geq Ca^{\beta}$ as $a \to 0$ and $\phi(t) = t^{\alpha}g(t)$ is an arbitrary gauge function, then $\int\limits_{0^+} \frac{g^{\beta+1}(s)}{s}ds = +\infty$ implies $P(\mathcal{P}^{\phi}(K(w)) = +\infty|K(w) \neq \emptyset) = 1$.

2. If $0 < r = \liminf_{a \to 0} -a^{-1/\beta} \log P(X < a)$, then for $\phi(t) = t^\alpha |\log|\log t||^\beta = t^\alpha g(t)$, $P(\mathcal{P}^\phi(K(w)) > 0 | K(w) \neq \emptyset) = 1$.

Does theorem 4.6 still hold if we only assume the random open set condition?

CHAPTER 5

Connection between random constructions and Galton-Watson trees

As mentioned before, there is a connection between Galton-Watson tree processes and random recursive constructions. Let N_σ, $\sigma \in \Delta^*$ be a sequence of i.i.d. random variables with non-negative integer values. The Galton-Watson tree T corresponding to this sequence is a subset of Δ^* such that $\emptyset \in T$ and $\sigma \in T \iff \sigma * i \in T$ for all $1 \le i \le N_\sigma$. The boundary, ∂T, of the random tree is the set of all infinite paths through the tree. The tree metric on ∂T is defined by setting for $\sigma, \tau \in \partial T$, $d_T(\sigma,\tau) = c^{|\sigma\wedge\tau|}$ when $\sigma \neq \tau$ and $d_T(\sigma,\tau) = 0$ if $\sigma = \tau$, where $c \in (0,1)$ and $\sigma \wedge \tau$ denotes the largest common subsequence of σ and τ. Liu ([19], [20]) has studied the dimension properties of ∂T with respect to the tree metric.

Suppose the random recursive construction satisfies $P(T_i = c|T_i \neq 0) = 1$. To simplify the matter relabel the cells on each level so that non-empty ones go first. Then a random map $\kappa_w\colon \partial T(w) \to K(w)$ can be considered, defined by $\sigma \mapsto \bigcap_k J_{\sigma|_k}$. If for some $\rho > 0$ for all σ and $i \neq j$ $P(\operatorname{dist}(J_{\sigma*i}, J_{\sigma*j}) \ge \rho\operatorname{diam}(J_\sigma)) = 1$, then κ_w is 1–1. The question arises as to the relationship between the tree metric on the limit set and the usual Euclidean metric from $\mathbb{R}^d$.

Proposition 5.1 *If for all σ $P(\exists x_\sigma \in J_\sigma \cap K(w)\colon \operatorname{dist}(x_\sigma, \partial J_\sigma) \ge \rho\operatorname{diam}(J_\sigma)) = 1$, then these two metrics are bi-Lipschitz equivalent.*

Proof: For all $x, y \in K(w)$ we obviously have $d(x,y) \le d_T(x,y)$. On the other hand, if there is a point inside each J_σ as in the condition of the proposition, then $d(x,y) \ge c\rho d_T(x,y)$. ■

We note that for the proofs of theorems about the packing and Hausdorff measures it suffices to have the second condition satisfied and $P(X = 1) < 1$, then they are valid (or invalid) for trees and random recursive constructions of this kind simultaneously (see [12], [19], [20]).

Liu ([20]) on pages 25–26 attempts to show that under certain conditions there exists the exact packing dimension for the branching process on a Galton-Watson tree when the number of offspring is at least 2. However the proof that the packing measure with respect to the gauge function is positive contains a mistake. We consider a gauge function $\phi(t) = t^\alpha g(t)$. By theorem 5 it is natural to assume that $\lim\limits_{t\to 0^+} g(t) = 0$, otherwise the corresponding packing measure will be infinite.

For a natural number $k > 3$ and $K > 0$ one constructs an antichain $\Gamma(w) = \Gamma_k(w) = \{\sigma \| \sigma| = k \text{ and for all } [\log k] \le j \le k-1 \; l^\alpha_{\sigma|_j} X_{\sigma|^*_j} > K\phi(l_{\sigma|_j})\} \cup \{\sigma | [\log k] \le |\sigma| \le k-1, l^\alpha_\sigma X_{\sigma^*} \le K\phi(l_\sigma) \text{ and for all } [\log k] \le j \le |\sigma|-1 \; l^\alpha_{\sigma|_j} X_{\sigma|^*_j} > K\phi(l_{\sigma|_j})\}$ where for $\sigma = (\sigma_1, \ldots, \sigma_{l-1}, \sigma_l)$, σ^* is the cyclic permutation of σ, given by

$$\sigma^* = \begin{cases} (\sigma_1, \ldots, \sigma_{l-1}, \sigma_l + 1), & \text{if } \sigma_l < n \\ (\sigma_1, \ldots, \sigma_{l-1}, 1), & \text{if } \sigma_l = n \end{cases}.$$

It is claimed that $E\Big[\sum\limits_{\sigma\in\Gamma(w)} l^\alpha_\sigma X_{\sigma^*}\Big] = E\Big[\sum\limits_{\sigma\in\Gamma(w)} l^\alpha_\sigma\Big] = 1$. But because the choice of $\Gamma(w)$ depends on X_{σ^*}, we have the following

Theorem 5.2 *For large k and $\Gamma(w) = \Gamma_k(w)$, $E\Big[\sum\limits_{\sigma\in\Gamma(w)} l^\alpha_\sigma X_{\sigma^*}\Big] < E\Big[\sum\limits_{\sigma\in\Gamma(w)} l^\alpha_\sigma\Big] = 1$. Moreover, in the proof of proposition 4.1 in [20], $\varliminf\limits_{k\to\infty} E\Big[\sum\limits_{\sigma\in\Gamma_k(w)} l^\alpha_\sigma X_{\sigma^*}\Big] = 0$.*

Proof: This can be seen as follows:

$$E\Big[\sum_{\sigma\in\Gamma(w)} l^\alpha_\sigma X_{\sigma^*}\Big] = \sum_{|\sigma|=k} E\Big[l^\alpha_\sigma X_{\sigma^*} \prod_{j=[\log k]}^{k-1} \mathbf{1}_{\{l^\alpha_{\sigma|_j} X_{\sigma|^*_j} > \phi(l_{\sigma|_j})\}}\Big] +$$

$$\sum_{l=[\log k]}^{k-1} \sum_{|\sigma|=l} E\Big[l^\alpha_\sigma X_{\sigma^*} \mathbf{1}_{\{l^\alpha_\sigma X_{\sigma^*} \le \phi(l_\sigma)\}} \prod_{j=[\log k]}^{l-1} \mathbf{1}_{\{l^\alpha_{\sigma|_j} X_{\sigma|^*_j} > \phi(l_{\sigma|_j})\}}\Big].$$

So, for $[\log k] \le l \le k-1$ we let

$$r_l = \sum_{|\sigma|=l} E\Big[X_{\sigma^*} \mathbf{1}_{\{l^\alpha_\sigma X_{\sigma^*} \le \phi(l_\sigma)\}} \prod_{j=[\log k]}^{l-1} \mathbf{1}_{\{l^\alpha_{\sigma|_j} X_{\sigma|^*_j} > \phi(l_{\sigma|_j})\}}\Big],$$

$$q_l = \sum_{|\sigma|=l} E\Big[\mathbf{1}_{\{l^\alpha_\sigma X_{\sigma^*} \le \phi(l_\sigma)\}} \prod_{j=[\log k]}^{l-1} \mathbf{1}_{\{l^\alpha_{\sigma|_j} X_{\sigma|^*_j} > \phi(l_{\sigma|_j})\}}\Big],$$

$$r_k = \sum_{|\sigma|=k} E\Big[X_{\sigma^*} \prod_{j=[\log k]}^{k-1} \mathbf{1}_{\{l^\alpha_{\sigma|_j} X_{\sigma|^*_j} > \phi(l_{\sigma|_j})\}}\Big], q_k = \sum_{|\sigma|=k} E\Big[\prod_{j=[\log k]}^{k-1} \mathbf{1}_{\{l^\alpha_{\sigma|_j} X_{\sigma|^*_j} > \phi(l_{\sigma|_j})\}}\Big].$$

Then the left-hand side of the inequality becomes $\sum\limits_{l=[\log k]}^{k} r_l$, and the right-hand side is $\sum\limits_{l=[\log k]}^{k} q_l$. By independence $q_k = r_k$. On the other hand for $[\log k] \le l \le k-1$ we have $r_l/q_l \le \sup\limits_{|\tau|=l} g(l_\tau) \to 0$ as $k \to \infty$. This gives $\lim\limits_{k\to\infty} \sum\limits_{l=[\log k]}^{k-1} r_l = 0$. Lines (4.3a)–(4.4), proposition 4.1 in Liu's paper yield $\liminf\limits_{k\to\infty} r_k = 0$. The result follows. ■

Therefore it remains unknown, if there is a gauge function in the exponential case (case 2 of theorem 4.11) that gives a.s. positive packing measure.

CHAPTER 6

Packing dimension of randomly generated distributions

There are several known ways for randomly generating probability measures on [0,1], see [11], [24]. In this chapter we prove that the packing dimension of a random probability measure generated by the scheme introduced in [24] coincides with its Hausdorff dimension almost surely. The *Hausdorff and packing dimensions of a probability measure* π on [0,1] are defined as follows:

$$\dim_H(\pi) = \inf\{\dim_H A | A \subset [0,1], \pi(A) = 1\},$$

$$\dim_P(\pi) = \inf\{\dim_P A | A \subset [0,1], \pi(A) = 1\}.$$

The setting is the same as in [24]. By $\mathcal{P}([0,1])$ we denote the set of all probability measures on [0,1]. Let τ be a mapping, or a transition kernel, from the dyadic rationals, $\mathcal{D}$, to $\mathcal{P}([0,1])$. Mauldin and Monticino define the *scaling map* $\theta : [0,1]^{\mathcal{D}} \to \mathcal{P}([0,1])$ as follows. Let $\mathcal{D}_n$ be the set of strictly n-th level dyadic rational, e.g. $1/2 \notin \mathcal{D}_2$. For $t = (t(1/2), t(1/4), t(3/4), \ldots) \in [0,1]^{\mathcal{D}}$, define the function $\theta(t)$ inductively on $\mathcal{D}$ by setting $\theta(t)(0) = 0,\ \theta(t)(1) = 1$. Asuming $\theta(t)|_{\bigcup\limits_{i=1}^{\infty} \mathcal{D}_i}$ has been defined, for $0 \le j \le 2^n - 1$ let

$$\theta(t)\Big(\frac{2j+1}{2^{n+1}}\Big) = \theta(t)\Big(\frac{j}{2^n}\Big) + \Big(\theta(t)\Big(\frac{j+1}{2^n}\Big) - \theta(t)\Big(\frac{j}{2^n}\Big)\Big) t\Big(\frac{2j+1}{2^{n+1}}\Big).$$

The probability neasure on the space of distribution functions, or prior, denoted by $\mathcal{R}_\tau$ is induced by $P_\tau = \prod\limits_{d \in \mathcal{D}} \tau(d)$ through θ.

Denote the distribution function of a probability measure π on [0,1] by h_π. A transition kernel $\tau : \mathcal{D} \to \mathcal{P}([0,1])$ is called centered, if, for each $\varepsilon > 0$, there exists a $\delta \in (0, 1/2)$ such that $\tau(d)(\delta, 1-\delta) > 1 - \varepsilon$ for all $d \in \mathcal{D}$. Let $\{0,1\}^*$ denote the set of all finite sequences of 0's and 1's including the empty sequence. We define a map $\beta : \{0,1\}^* \to \mathcal{D}$ by $\beta(\emptyset) = 1/2$, and for all other $(b_1, \ldots, b_n) \in \{0,1\}^*$,

$\beta((b_1,\ldots,b_n)) = 1/2 - \sum_{i=1}^{n}(-1)^{b_i}/2^{i+1}$. For a transition kernel $\tau : \mathcal{D} \to \mathcal{P}([0,1])$, set $\tau^* = \tau\beta : \{0,1\}^* \to \mathcal{P}([0,1])$. For $x \in [0,1)$ and $n \geq 0$, define $\alpha_n(x) = i/2^n \leq x < (i+1)/2^n$. And if $0.b_1b_2\ldots b_n$ is the dyadic expansion of $i/2^n$, let $b_0(i/2^n) = \emptyset$, and for $1 \leq k \leq n$, let $b_k(i/2^n) = (b_1,\ldots,b_k)$. For a transition kernel τ and $b \in \{0,1\}^*$, set $\gamma_b = \int_{[0,1]} y\log(y) + (1-y)\log(1-y)d\tau^*(b)(y)$, and for $n \geq 1$, let $\sigma_n^2 = \sup_{b\in\{0,1\}^*}\Big\{\int y(\log(y)-\gamma_b)^2 + (1-y)(\log(1-y)-\gamma_b)^2 d\tau^*(b)(y)\Big\}$.

Lemma 6.1 *Suppose that $\pi \in \mathcal{P}([0,1])$ is continuous and*

$$\lim_{n\to\infty}\frac{1}{n}\Big(\log\Big(h_\pi(\alpha_n(x)+1/2^n) - h_\pi(\alpha_n(x))\Big)\Big) = \gamma_x$$

for h_π-almost all $x \in [0,1]$. If $\gamma_1 = \inf\{\gamma_x | x \in [0,1]\}$, then $\dim_P(\pi) \leq -\gamma_1/\log 2$.

Proof: Let $\eta = -\gamma_1/\log 2 + \varepsilon$. Since $\lim_{n\to\infty}\frac{1}{n}\Big(\log\Big(h_\pi(\alpha_n(x)+1/2^n) - h_\pi(\alpha_n(x))\Big)\Big) = \gamma_x \geq \gamma_1$, then $\lim_{n\to\infty}\Big(h_\pi(\alpha_n(x)+1/2^n) - h_\pi(\alpha_n(x))\Big)^{1/n} \geq e^{\gamma_1}$, and $\lim_{n\to\infty}\Big(h_\pi(\alpha_n(x)+1/2^n) - h_\pi(\alpha_n(x))\Big)2^{-n\gamma_1/\log 2} \geq 1$. Note that for all $x \in [0,1]$ and $0 < r < 1$, if $2^{-k+1} > r \geq 2^{-k}$, then $x + r \geq \alpha_k(x) + 1/2^k$ and $x - r \leq \alpha_k(x)$. Therefore for h_π-almost every $x \in [0,1]$,

$$\underline{\lim}_{r\to 0}\frac{\pi(B(x,r))}{r^\eta} = \underline{\lim}_{r\to 0}\frac{h_\pi(x+r) - h_\pi(x-r)}{r^\eta} \geq \underline{\lim}_{n\to\infty}\frac{h_\pi(\alpha_n(x)+1/2^n) - h_\pi(\alpha_n(x))}{2^{-n\eta}}$$

$$= \underline{\lim}_{n\to\infty} h_\pi(\alpha_n(x)+1/2^n) - h_\pi(\alpha_n(x))2^{-n\gamma_1/\log 2}2^{n\varepsilon} = \infty.$$

Thus by the density theorem we see that there exists $A \subset [0,1]$ with $\pi(A) = 1$ such that $\mathcal{P}^\eta(A) = 0$, i.e. $\dim_P A \leq \eta$. Thus $\dim_P(\pi) \leq \eta$. The result follows. ∎

The next lemma is a slight modification of what is contained in the proof of theorem 5.2,[24].

Lemma 6.2 *Suppose that the condition of lemma 6.1 holds and let $\gamma < 0$, $C = \{x \in [0,1] | \gamma_x < \gamma\}$. If $\pi(C) > 0$, then $\dim_H C \geq -\gamma/\log 2$.*

Proof: The proof of theorem 5.12 in [24] can be applied almost without change. We adduce it for the reader's convenience. Fix $\varepsilon > 0$. For $\delta > 0$ and $x \in C$, let $m(x)$ be the smallest n (for x where such n exists) such that $2^{-n} < \delta$ and $h_\pi(\alpha_n(x)+1/2^n) - h_\pi(\alpha_n(x)) > (1/2^n)^{-\gamma/\log 2-\varepsilon}$. Set $K(\varepsilon,\delta) = \Big\{\Big[\alpha_{m(x)}(x), \alpha_{m(x)}(x)+1/2^{m(x)}\Big]\Big\}_{x\in C}$. Then $0 = \pi\Big(\bigcap_{\delta\to 0} K(\varepsilon,\delta)\Big) = \lim_{\delta\to 0}\pi(K(\varepsilon,\delta))$ and there exists $f(\varepsilon,\delta)$ such that $\pi((I\cap C)\setminus K(\varepsilon,\delta)) < |I|^{(-\gamma/\log 2)-f(\varepsilon,\delta)}$ for any interval I for which $|I| < \delta$. Moreover, $f(\varepsilon,\delta)$ can be chosen so that $f(\varepsilon,\delta)$ decreases (to ε) as δ decreases and $f(\varepsilon,\delta)\to 0$ as ε and $\delta\to 0$.

Now fix $\hat\delta > 0$. We will show that $\mathcal{H}^\kappa_\delta(C) \geq \pi(C)/2$, where $\kappa = -\gamma/\log 2 - f(\varepsilon,\hat\delta)$. Let $\{I_n\}$ be a set of intervals covering $C\setminus K(\varepsilon,\hat\delta)$ with $|I_n| < \delta < \hat\delta$ and $\pi(K(\varepsilon,\hat\delta)) < \pi(C)/2$. Then

$$\pi(C)/2 \leq \pi(C\setminus K(\varepsilon,\hat\delta)) \leq \sum_n \pi((I_n\cap C)\setminus K(\varepsilon,\delta)) \leq \sum_n |I_n|^{(-\gamma/\log 2)-f(\varepsilon,\hat\delta)}.$$

Hence,

$$\pi(C)/2 \leq \mathcal{H}^{-\gamma/\log 2-f(\varepsilon,\hat\delta)}(C\setminus K(\varepsilon,\hat\delta)) \leq \mathcal{H}^{-\gamma/\log 2-f(\varepsilon,\hat\delta)}(C).$$

Since $f(\varepsilon,\delta)$ can be arbitrarily small, it follows that $\dim_H C \geq -\gamma/\log 2$. ■

Theorem 6.3 *Let $\tau : \mathcal{D}\to\mathcal{P}([0,1])$ be a centered transition kernel. Suppose*

i. $\int_{[0,1]} |y\log y + (1-y)\log(1-y)|d\tau(d)(y) < \infty$ *for all* $d\in\mathcal{D}$,

ii. $\sum_{n=1}^{\infty} \sigma_n^2/n^2 < \infty$, *and*

iii. $\gamma_x = \lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\gamma_{b_j(\alpha_n(x))}$ *exists for all* $x\in[0,1]$.

Then for R_τ-almost all π, $\dim_P(\pi) = \dim_H(\pi)$.

Proof: Let $A = \Big\{\pi\in\mathcal{P}([0,1])|\pi$ is continuous and $\lim_{n\to\infty}\frac{1}{n}\log\Big(h_\pi(\alpha_n(x)+1/2^n) - h_\pi(\alpha_n(x))\Big) = \gamma_x$ for h_π-almost all $x\in[0,1]\Big\}$. By lemma 5.12 in [24], $R_\pi(A) = 1$. Suppose that for some $\pi\in A$, there exists d such that $\dim_H(\pi) < d < \dim_P(\pi)$.

Then there exists $B \subset [0,1]$ with $\pi(B) = 1$ such that $\mathcal{H}^d(B) = 0$ and $\mathcal{P}^d(B) = \infty$. By lemma 6.1, $\gamma_1 = \inf\{\gamma_x | x \in [0,1]\} < -d \log 2$. By lemma 6.2, $\pi(\{x \in [0,1] | \gamma_x < -d \log 2\}) = 0$, which contradicts the continuity of π. ∎

The Hausdorff dimension (as well as packing dimension) of a random distribution function generated according to the above scheme remains unknown. There are only some estimates on it that can be found in [24].

CHAPTER 7

Examples

Example 7.1 *Mandelbrot percolation or canonical curdling.*

Mandelbrot introduced the following process which he termed canonical curdling. Fix an integer $n > 1$ and a number p with $0 < p < 1$. Partition the unit square into n^2 congruent subsquares. Let each subsquare survive independently with probability p. For each subsquare which survives repeat the process. This is a n^2-ary random recursive construction. The limit set is nonempty with positive probability provided $p > 1/n^2$. The Hausdorff dimension in this case is $\alpha = 2 + (\log p / \log n)$. The exact Hausdorff gauge function is $t^\alpha(|\log|\log t||)^{1-(\alpha/2)}$ as determined by Graf et al.([12]) in Example 6.2. By theorem 3.2, the limit set of the Mandelbrot percolation process provided it is nonempty also has packing dimension $\alpha = 2 + \log p / \log n$ a.s. This also follows from the results of Gatzouras and Lalley ([9]). By theorem 4.6 its α-packing measure is infinite. It is known (see [2]) that $P(X < a) \asymp a^\beta$ as $a \to 0$ where β satisfies $p_1 m^\beta = 1$, $p_1 = P(\exists! i \colon T_i \neq 0) = n^2 p(1-p)^{n^2-1}$ and $m = n^2 p$ is the expected number of offspring. In this case, $\beta = -1 - \frac{\log(1-p)^{n^2-1}}{\log n^2 p}$. Hence from theorem 4.11, part 1 we deduce that for the gauge function $\phi(t) = t^\alpha g(t)$ such that $\int_{0+} \frac{g(s)^{\beta+1} ds}{s} < +\infty$, $\mathcal{P}^\phi(K(w)) = 0$. We conjecture that as in the article of Taylor ([32]) there is no exact packing measure function.

Example 7.2 *The zero set of the Brownian bridge.*

Graf et al.([12]) have shown that this set can be represented as a random recursive construction and the distribution density of the vector (T_1, T_2) has been found. The Hausdorff dimension of this set is known to be 1/2. Therefore its packing and box-counting dimensions are 1/2, and packing measure in dimension 1/2 is infinite.

Using the distribution density it is easy to show that $P(T_{1,2} < a) = O(\sqrt{a}), a \to 0$. By a result of Liu ([21]), we obtain that $P(X < a) = O(a), a \to 0$. Graf et al.([12]) in example 6.1 show that the condition of theorem 4.11 is satisfied, therefore $\int\limits_{0^+} \frac{g^2(s)}{s} ds < +\infty$ implies $P(\mathcal{P}^\phi(K(w)) = 0 | K(w) \neq \emptyset) = 1$, and our hypothesis would say that $\int\limits_{0^+} \frac{g^2(s)}{s} ds = +\infty$ implies $P(\mathcal{P}^\phi(K(w)) = +\infty | K(w) \neq \emptyset) = 1$. This is actually proven by Feng and Sha ([6]) from the view point of subordinators.

Example 7.3 *A random Cantor set.*

Choose two numbers independently with respect to the uniform distribution on $J_\emptyset = [0, 1]$. J_1 is the left most interval and J_2 is the right most interval in the partition of $J_\emptyset$ thus obtained. Its Hausdorff dimension α has been found to be $(\sqrt{17} - 3)/2$, and the exact Hausdorff dimension function is $t^\alpha |\log|\log t||^{1-\alpha}$ (see [12], [27]). By theorem 3.2, it has the same packing and box-counting dimensions and by theorem 4.6, its packing measure in dimension $(\sqrt{17} - 3)/2$ is infinite.

One can calculate $P(T_1 < a) = P(T_2 < a) = 2a - a^2$. Hence, $P(T_{1,2} < a) = O(a), a \to 0$ and again according to Liu ([21]) $P(X < a) = O(a^2), a \to 0$. Therefore by theorem 4.11, $\int\limits_{0^+} \frac{g^3(s)}{s} ds < +\infty$ implies $P(\mathcal{P}^\phi(K(w)) = 0) = 1$ and our hypothesis would say that $\int\limits_{0^+} \frac{g^3(s)}{s} ds = +\infty$ implies $P(\mathcal{P}^\phi(K(w)) = +\infty) = 1$.

Example 7.4 *Modified Mandelbrot percolation or modified curdling.*

This process was proposed by Dekking and Grimmett. It was discussed in detail by Graf et al.([12]) in example 6.12 and they found the exact Hausdorff gauge function for this construction. Fix a positive integer $n > 1$ and a probability measure μ on the power set of $\{1, ..., n^2\}$. Let $J_1, ..., J_{n^2}$ be a labeling of the partition of $[0, 1] \times [0, 1]$ into congruent subsquares. If the square J_σ has been constructed, then choose $A \subset \{1, ..., n^2\}$ according to μ and let $J_{\sigma * i}, i \in A$ be the subsquares of J_σ obtained by scaling J_i into J_σ via the natural map. If m is the average number of offspring, then $\alpha = \log mp / \log n$. If $\underline{m}$, the essential infimum of the number of offspring, is at least 2, then according to Liu ([20], 2.3a), the second case in theorem 4.11 holds, and for $\beta = 1 - \log m / \log \underline{m}$, the gauge function $\phi(t) = t^\alpha |\log|\log t||^\beta$, we have $\mathcal{P}^\phi(K(w)) < \infty$ a.s. We conjecture that it is positive a.s.

Example 7.5 *Random recursive construction for which* $\overline{\dim}_B K$ *is a non-degenerate random variable and* $\dim_H K < \dim_P K < \operatorname{ess\,inf} \overline{\dim}_B K$ *a.s.*

Recall from [4] that for $p > 0$, $\overline{\dim}_B\{1/n^p, n \in \mathbb{N}\} = \frac{1}{p+1}$. Let $J = [0,1]$ and take p with respect to the uniform distribution on $[1,2]$. We build a random recursive construction so that on level 1, the right endpoints of offspring are the points $1/n^p$, $n \in \mathbb{N}$. On all other levels, the offspring are formed from a scaled copy of $[0,1]$ and its disjoint subintervals with right endpoints at $1/n^4$, $n \in \mathbb{N}$. Let $(V_1, V_2, \dots)$ be a fixed vector of reduction ratios so that $V_n = (1/1024)^n \inf_{1 \le p \le 4}\{1/i^p - 1/(i+1)^p\}$. Then $\sum_{n=1}^{\infty} V_n^{1/8} < 1$, $K(w) \neq \emptyset$, $\dim_H K \le 1/8$ and $\overline{\dim}_B K = \max\{\dim_H K, \frac{1}{p+1}\} = \frac{1}{p+1}$, where p is chosen according to the uniform distribution on $[1,2]$. Hence, $\operatorname{ess\,inf} \overline{\dim}_B K = 1/3$. By theorem 3.11, $\dim_P K = 1/5$.

Example 7.6 *Random recursive construction for which the zero-one law does not hold.*

Let $J = [0,1]$ and take $p(w), w \in \Omega$ with respect to the uniform distribution on $[1,2]$. We build a random recursive construction so that on level 1, the right endpoints of offspring are the points $1/n^p$, $n \in \mathbb{N}$, and the length of the n-th offspring is $V_n = (1/16^n) \inf_{1 \le p \le 2}\{1/n^p - 1/(n+1)^p\}$. On all other levels, the offspring are formed from a scaled copy of $[0,1]$ and its disjoint subintervals of length V_n with right endpoints at $1/n^p$, $n \in \mathbb{N}$. Obviously, $\sum_{n=1}^{\infty} V_n^{1/4} < \infty$, and hence for each $w \in \Omega$, we have $\dim_H K \le 1/4$. On the other hand we can use the results from [26] to determine that for each $w \in \Omega$, $\dim_P K(w) = \overline{\dim}_B K(w) = \frac{1}{p(w)+1}$. So, the reduction ratios are constant, but random placement of the offspring gives non-trivial variation of the packing dimension.

Example 7.7 *Random self-avoiding process on the Sierpinski gasket.*

This process was introduced in [14], and its almost sure Hausdorff dimension was found in [13]. Here we give a simpler alternative definition of it. We prove that it is a limit set of a random recursive construction which allows to apply already known theorems to find its dimensions.

Let J be an equilateral triangle of diameter 1 with one vertex O at the origin and another vertex B at a point with coordinates (1,0). By A we denote the third vertex of this triangle. J_1, J_2, J_3 are those three equilateral triangles of diameter 1/2 out of 4 partitioning J that have as one of their vertices O, A or B correspondingly. Then the process is iterated, and we obtain a (non-random) self-similar set which is called *the Sierpinski Gasket*, $G = \bigcap_{n=1}^{\infty} \bigcup_{\sigma\in\{1,2,3\}^n} J_\sigma$.

Fix $1 > p > 0$. Let $f_n(x) : [0,1] \to \bigcup_{\sigma\in\{1,2,3\}^n} J_\sigma$ be a collection of random maps such that for all $\sigma \in \{1,2,3\}^n$, $J_\sigma \cap f_n([0,1])$ coincides with a side of triangle J_σ or is empty in the following way:

i. For $n = 0$, $f_0(0) = O$, $f_0(1) = A$, and the map f_0 is linear.

ii. Suppose that the random function f_n has been defined. For a fixed $\sigma \in \{1,2,3\}^n$, let $[a_\sigma, b_\sigma] = f_n^{-1}(J_\sigma \cap f_n([0,1]))$. Let $m \in \{1,2,3\}$ be such that $J_{\sigma*m} \cap J_\sigma \cap f_n([0,1]) = \emptyset$, $k \in \{1,2,3\}$ such that $f(a_\sigma) \in J_{\sigma*k}$ and l such that $f(b_\sigma) \in J_{\sigma*l}$. Define f_{n+1} so that $f_{n+1}(a_\sigma) = f_n(a_\sigma)$, $f_{n+1}(b_\sigma) = f_n(b_\sigma)$. With probability p, we let $f_{n+1}((a_\sigma+b_\sigma)/2) = J_{\sigma*k} \cap J_{\sigma*l}$, and with probability $1-p$, $f_{n+1}((a_\sigma+b_\sigma)/3) = J_{\sigma*k} \cap J_{\sigma*m}$ and $f_{n+1}(2(a_\sigma+b_\sigma)/3) = J_{\sigma*m} \cap J_{\sigma*l}$. Then the map f_{n+1} is extended by linearity. Inside all J_σ's, the process of refining of f_n to f_{n+1} is independent.

Finally we define a random map $f : [0,1] \to G$ by setting $f(x) = \lim_{n\to\infty} f_n(x)$. Function f is well defined because for all n and $x \in [0,1]$, $\operatorname{dist}(f_n(x), f_{n+1}(x)) \le 2^{-n}$, and hence the limit exists, moreover, f is continuous as a uniform limit of continuous functions. Note that for all n, $f_n(0) = O$ and $f_n(1) = A$, therefore $f(0) = O$ and $f(1) = A$. Also for all $x \in [0,1]$, the $\operatorname{dist}(f_n(x), G) \le 2^{-n}$, and since G is compact, we obtain that $f([0,1]) \subset G$.

Now we show that f is a one-to-one function. Suppose that $0 \le a < b \le 1$. Suppose that $f(a) = f(b) = D$. Let $A_n = \cup\{J_\sigma | \sigma \in \{1,2,3\}^n, f_n(a) \in J_\sigma, J_\sigma \cap f_n([0,1]) \ne \emptyset\}$, $B_n = \cup\{J_\sigma | \sigma \in \{1,2,3\}^n, f_n(b) \in J_\sigma, J_\sigma \cap f_n([0,1]) \ne \emptyset\}$. By definition of the sequence f_n, if $f_n(a) \in A_n$, then for all $k \ge n$, $f_k(a) \in A_n$ and $A_k \subset A_n$. Since A_n is closed, $D = \lim_{k\to\infty} f_k(a) \in A_n$. Any two triangles J_σ with $|\sigma| = n$ can have at most

1 common point, therefore A_n can include at most 2 triangles and $\mathrm{diam}(A_n) \le 2^{-n+1}$. Thus we obtain that $D = \bigcap_{n=1}^{\infty} A_n$. Similarly we obtain that $D = \bigcap_{n=1}^{\infty} B_n$. Since A_n and B_n are two decreasing sequences of closed sets, $A_n \cap B_n \neq \emptyset$ for all n, which is possible if and only if $A_n \cup B_n$ consists of at most 2 triangles having a common point. However, if $k > |\log_3(b-a)| + 2$, then $f_k(a)$ and $f_k(b)$ must be in two disjoint triangles because for all $\sigma \in \{1,2,3\}^k$, either $J_\sigma \cap f_k([0,1])$ is an edge of J_σ, or $J_\sigma \cap f_k([0,1]) = \emptyset$, and once the path $f_k([0,1])$ leaves triangle J_σ, it never enters this triangle again. This is a contradiction, and thus $f([0,1])$ is a random arc.

Now we redefine the triangles J_σ such that for each σ, if only two triangles out of $J_{\sigma*1}$, $J_{\sigma*2}$, $J_{\sigma*3}$ intersect with $f_{|\sigma|+1}([0,1])$ along an edge, then these two triangles will be denoted by $J_{\sigma*1}$ and $J_{\sigma*2}$, and $J_{\sigma*3} = \emptyset$. Thus for each σ, the random vector of reduction ratios is $(1/2, 1/2, 0)$ with probability p, and $(1/2, 1/2, 1/2)$ with probability $1-p$. Then for all $k \ge n$, $f_k([0,1]) \subset \bigcup_{|\sigma|=n} J_\sigma$ and hence $f([0,1])$ is a subset of the limit set of a random recursive construction formed by the sets J_σ. If $y \in f([0,1]) \setminus \bigcap_{n=1}^{\infty} \bigcup_{|\sigma|=n} J_\sigma$, then there exists $k \in \mathbb{N}$ such that $\mathrm{dist}(y, \bigcap_{n=1}^{\infty} \bigcup_{|\sigma|=n} J_\sigma) > 2^{-k+1}$ and therefore $\mathrm{dist}(y, \bigcup_{|\sigma|=k} J_\sigma) > 2^{-k}$. On the other hand, $\mathrm{dist}(y, \bigcup_{|\sigma|=k} J_\sigma) \le \mathrm{dist}(y, f_k(f^{-1}(y))) + \mathrm{dist}(f_k(f^{-1}(y)), \bigcup_{|\sigma|=k} J_\sigma) \le 2^{-k} + 0 = 2^{-k}$. This contradiction shows that $f([0,1])$ coincides with the limit set.

By theorem 3.2 the Hausdorff, packing and Minkowski dimensions of $f([0,1])$, $\alpha = \log_2(3-p)$ almost surely. By theorem 4.11, for the gauge function $\phi(t) = t^\alpha |\log|\log t||^{1-\alpha}$, $\mathcal{P}^\phi(f[0,1]) < \infty$ a.s.

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